DOI: 10.3842/nosc.v28i1.1496

UDC 517.9

# QUALITATIVE ANALYSIS OF SOLUTIONS TO LIENARD STOCHASTIC DIFFERENTIAL EQUATION WITH MULTIPLE DELAYS

# ЯКІСНИЙ АНАЛІЗ РОЗВ'ЯЗКІВ СТОХАСТИЧНОГО ДИФЕРЕНЦІАЛЬНОГО РІВНЯННЯ ЛІЕНАРДА З КРАТНИМИ ЗАПІЗНЕННЯМИ

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This study focuses on providing criteria for stochastic stability of the zero solution and uniform boundedness of solutions to a class of nonlinear Lienard stochastic differential equations with multiple bounded delays. A Lyapunov – Krasovskii functional is constructed and employed as a tool to prove our results. By our results, many stability and boundedness theorems of second order are improved on and are also generalized. The credibility of our results is demonstrated by two numerical examples included.

Встановлено критерії стохастичної стійкості нульового розв'язку та рівномірної обмеженості розв'язків класу нелінійних стохастичних диференціальних рівнянь Ліенарда з кратними обмеженими запізненнями. Побудовано функціонал Ляпунова – Красовського, який використовується як інструмент для доведення шуканих результатів. За допомогою цих результатів покращено та узагальнено багато теорем про стійкість й обмеженість другого порядку. Достовірність отриманих результатів демонструють два наведені числові приклади.

**1. Introduction.** A differential equation involving a stochastic process is termed a stochastic differential equations (SDE) and the solution of such equation is also a stochastic process. In this work, we shall give attention to the study of stochastic stability (SS) and uniform boundedness (UB) theorems of the following Lienard scalar SDE of type:

$$x'' + e_1(t)f\big(t,x,x'\big)x' + e_2(t)g\big(t,x,x'\big) + \sum_{i=1}^{n} k_i(t)h_i(x(t-r_i(t))) + e_3(t,x(t))\eta'(t) = m\big(t,x,x'\big), \tag{1}$$

where $f, g, m \in C(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $e_3 \in C(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R})$, $e_1, e_2, k_i, r_i \in C(\mathbb{R}^+, \mathbb{R})$, $h_i \in C(\mathbb{R}, \mathbb{R})$, the delay term $r_i(t)$ is continuous such that $0 \leq r(t) = \max_{1\leq i\leq n} r_i(t) \leq \lambda$, ($\lambda > 0$, is a constant whose value is to be specified later) and $r_i'(t) \leq \kappa_1,\ 0 < \kappa_1 < 1$, and functions $g(0) = h_i(0) = 0$ for all $i = 1, 2, 3, \ldots, n$. Lastly, we assumed that the functions contained in (1) satisfy local Lipschitz condition in their respective arguments to have a unique global solution, $\mathbb{R}^+ = [0, \infty)$, $\mathbb{R} = (-\infty, +\infty)$, $\eta(t) \in \mathbb{R}$ is a standard Brownian motion and the $'$ denotes differentiation with respect to time $t$.

It is interesting to note that there have been many wonderful results on the qualitative study of solutions of ordinary, stochastic and functional differential equations in the literature (see, e.g., [1 – 20]). Our findings also show that the second method of Lyapunov was employed in all of these cited results. The main idea of this method depends on constructing an energy function which is nonnegative everywhere save at the origin where it is zero while its derivative is negative semi-definite. Having this function in place, one can easily discuss stability, boundedness and other qualitative properties of any differential equation without solving the equation itself. Lienard differential equations as well as other differential equations are generally useful in modeling population growth, spread of diseases, prey and predator interaction, development of atomic bomb and many other applications [21 – 27].

A careful review of literature reveals that asymptotic stability property of zero solution to

$$x''(t) + ax'(t) + bx(t-h) + \sigma x(t)\omega'(t) = 0,$$

and

$$x''(t) + ax'(t) + f(x(t-h)) + \sigma x(t-\tau)\omega'(t) = 0,$$

where $h$ and $\tau$ are fixed delay and $\omega$ is the Wiener process, was proved in [1]. While in 2023, Mahmoud et al. [13] used a Lyapunov – Krasovskii functional (LKF) to provide certain conditions that ensure stability and boundedness of solution to

$$x'' + \phi(t)f\big(x, x'\big)x' + \sum_{i=1}^{n} g_i(x) + \sum_{i=1}^{n} \psi_i(t)h_i(x(t-\tau_i(t))) + \Delta\vartheta'(t) = e\big(t, x, x'\big),$$

such that $\Delta$ is a positive constant, $\tau_i$ are bounded delays and $\vartheta$ is a Brownian process. In another recent article [28], the authors discussed SS and UB of solutions to

$$x'' + g\big(x, x'\big)x' + \sum_{i=1}^{n} b_i(t)h_i(x) + \sum_{i=1}^{n} f_i(x(t-\tau_i)) + f(t,x)\omega'(t) = r\big(t, x, x', x(t-\tau_0), x'(t-\tau_0)\big),$$

where $\tau = \max_{0\leq i\leq n} \tau_i,\ t \in [-\tau, 0]$, and $\omega(t) \in \mathbb{R}$ is a Brownian process. The results were established using a LKF as a tool. This current research is motivated by the recent results in [13] and [28] and our goal is to provide stability and boundedness criteria to a more general Lienard equation (1).

**2. Some definitions and basic results.** Here, we introduce some definitions and standard results relevant to the study. Without any misconception, we will write $\Gamma(t, x(t))$, $\Omega(t, x(t))$, $\Theta(t, x(t))$ as $\Gamma(,)$, $\Omega(,)$, $\Theta(,)$ respectively where necessary.

Let us consider the following general form of a scalar SDE:

$$dx(t) = \Gamma(,)dt + \Omega(,))d\Lambda(t), \tag{2}$$

defined for $t \geq t_0$ and $x(t_0) = x_0$, where both $\Gamma(,)$, $\Omega(,) : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ and $\Lambda(t)$ is the one-dimensional Wiener (stochastic) process. Given that both $\Gamma$ and $\Omega$ are sufficiently smooth enough such that (2) has a unique continuous solution on $t \geq 0$ which is denoted $x(t, x_0)$. Also, if $x_0 = 0$ and $\Gamma(t,0) = \Omega(t,0) = 0$ for all $t \geq 0$. Then (2) has a zero solution $x(t,0) \equiv 0$.

**Definition 2.1.** *Suppose that $\Lambda(t)$ is a standard Brownian motion on a probability space $(\omega, \mathcal{F}, P)$ and let $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ be the completion of the minimal filtration by null set. A strong solution of the SDE (2) with initial condition $x_0 \in \mathbb{R}$ is an adapted process $x_t = x_t^{x_0}$ with continuous paths such that for all $t \geq 0$,*

$$x_t = x_0 + \int_0^t \Gamma(,)\, ds + \int_0^t \Omega(,))\, d\Lambda(s) \quad a.s.$$

**Definition 2.2.** *The zero solution of the SDE (2) is said to be stochastically stable or stable in probability, if for every pair of $\epsilon \in (0,1)$ and $\eth > 0$, there exists a $\delta_0 = \delta_0(\epsilon, \eth) > 0$ such that $Pr\{|x(t; x(0))| < \eth \ \forall t \geq 0\} \geq 1 - \epsilon$, whenever $|x(0)| < \delta_0$.*

**Definition 2.3.** *The zero solution of the SDE (2) is said to be uniformly stochastically asymptotically stable, if it is stochastically stable and in addition if for every pair of $\epsilon \in (0,1)$ and $\eth > 0$, there exists a $\delta_0 > 0$ such that $Pr\{\lim_{t\to\infty} x(t; x(0)) = 0\} \geq 1 - \epsilon$, whenever $|x(0)| < \delta_0$.*

**Definition 2.4.** *A solution $x(t, x(0))$ of the SDE (2) is said to be stochastically bounded or bounded in probability if it satisfies*

$$E^{x(0)} \|x(t, x(0))\| \leq B(t_0, \|x(0)\|) \quad \forall t \geq t_0, \tag{3}$$

*where $E^{x(0)}$ denotes the expectation operator with respect to the probability law associated with $x(0)$, $B : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a constant depending on $t_0$ and $x(0)$.*

**Definition 2.5.** *The solution $x(t_0, x(0))$ of the SDE (2) is said to be uniformly stochastically bounded, if $B$ in (3) is independent of $t_0$.*

Let $\Theta(,)$ be a family of nonnegative continuous functions (Lyapunov functions) of $t \geq 0$ and $x(t) \in \mathbb{R}$, so that $x(t)$ is a solution of (2). It is required that $\Theta(,)$ be one time differentiable in $t$ and twice differentiable in $x(t)$. Thus, according to Itô formula, we have

$$d\Theta(,) = L\Theta(,)dt + \Theta_x(,)\Omega(,)d\Lambda(t), \tag{4}$$

where

$$L\Theta(,) = \Theta_t(,) + \Theta_x(,)\Gamma(,) + \frac{1}{2}\operatorname{trace}\big[\Omega^T(,)\Theta_{xx}(,)\Omega(,)\big], \tag{5}$$

$\Theta_x$ and $\Theta_{xx}$ are respectively the first and second partial derivatives of $\Theta(,)$ with respect to $x$.

**Lemma 2.1** [29]. *Let there be $\Theta(,) \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+)$, and certain positive constants $q_1$, $q_2$ and $q_3$ so that:*

(i) $\Theta(t,0) = 0$*;*

(ii) $q_1(\|x\|) \leq \Theta(,) \leq q_2(\|x\|), q_1(u) \to \infty$ *as* $u \to \infty$*;*

(iii) $L\Theta(,) \leq -q_3(\|x\|)$ *for all* $(t,x)$.

*Then the zero solution of (2) is uniformly stochastically asymptotically stable in the large.*

***Assumption 2.1*** [29, 30]. *Let* $\Theta(,) \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+)$, *and assume that for any solution* $x(t_0, x_0)$ *of (2) and any fixed* $0 \le t_0 \le T < \infty$, *we have*

$$E^{x_0}\left\{\int\limits_{t_0}^{T} \Theta^2_{x_i}(,)\Omega^2_{ik}(,)\,dt\right\} < \infty, \quad 1 \le i \le n, \quad 1 \le k \le m. \tag{6}$$

***Assumption 2.2*** [29, 30]. *A particular case of the general inequality (6) is the following condition. Assume that there exists a function* $p(t)$ *such that*

$$|\Theta_{x_i}(,)\,\Omega_{ik}(,)| < p(t), \tag{7}$$

*where* $x \in \mathbb{R}^2,\ 1 \le i \le n,\ 1 \le k \le m,$ *and for any fixed* $0 \le t_0 \le T < \infty$,

$$\int\limits_{t_0}^{T} p^2(t)\,dt < \infty. \tag{8}$$

**Lemma 2.2** [29, 30]. *Suppose that there exits a Lyapunov function* $\Theta(,) \in C^{1,2} = (\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+)$, *satisfying Assumption 2.1, such that, for all* $(t, x(t)) \in \mathbb{R}^+ \times \mathbb{R}^n$,

(i) $\|x(t)\|^j \le \Theta(,)$;

(ii) $L\Theta(,) \le -\alpha(t)\|x(t)\|^k + \psi(t)$;

(iii) $\Theta(,) - \Theta^k(,) \le \mu$,

*where* $\alpha, \psi \in C(\mathbb{R}^+, \mathbb{R}^+)$, $j$ *and* $k$ *are positive constants,* $j \ge 1$ *and* $\mu$ *is a nonnegative constant. Then the solutions of (2) satisfy*

$$E^{x_0}\|x(t, x_0)\| \le \left[\Theta(t_0, x_0)e^{-\int_{t_0}^{t}\alpha(\varepsilon)d\varepsilon} + \int\limits_{t_0}^{t}(\mu\alpha(u) + \psi(u))e^{-\int_{u}^{t}\alpha(\varepsilon)d\varepsilon}\,du\right]^{1/j} \quad \forall t \ge t_0. \tag{9}$$

**Lemma 2.3** [29, 30]. *Assume that there exist a Lyapunov function* $\Theta(,) \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+)$, *satisfying Assumption 2.2, such that, for all* $(t, x(t)) \in \mathbb{R}^+ \times \mathbb{R}^n$,

(i) $\|x(t)\|^j \le \Theta(,) \le \|x(t)\|^k$;

(ii) $L\Theta(,) \le -\alpha(t)\|x(t)\|^{\overline{\xi}} + \psi(t)$;

(iii) $\Theta(,) - \Theta^{\overline{\xi}/k}(,) \le \mu$,

*where* $\alpha, \psi \in C(\mathbb{R}^+, \mathbb{R}^+)$, $j$, $k$, *and* $\overline{\xi}$ *are positive constants,* $j \ge 1$ *and* $\mu$ *is a nonnegative constant. Then the solutions of (2) satisfy inequality (9) for all* $t \ge t_0$.

***Corollary 2.1*** [29, 30].

(i) *Suppose that all the hypotheses of Lemma 2.2 hold and, in addition,*

$$\int\limits_{t_0}^{t}(\mu\alpha(u) + \psi(u))e^{-\int_{u}^{t}\alpha(\varepsilon)d\varepsilon}\,du \le B, \tag{10}$$

*for all* $t \ge t_0 \ge 0$ *and some positive constant* $B$*; then all solutions of (2) are stochastically bounded.*

(ii) *Again suppose that all the hypotheses of Lemma 2.3 hold and, in addition, if condition (10) is satisfied, then all solutions of (2) are uniformly stochastically bounded.*

For more details on this section, interested individual can check [1, 29].

**3. Main results.** Let us use $x' = y$ to write (1) as

$$\begin{aligned} x' &= y, \\ y' &= -e_1(t)f(t,x,y)y - e_2(t)g(t,x,y) - \sum_{i=0}^{n} k_i(t)h_i(x) \\ &\quad + \sum_{i=0}^{n} k_i(t) \int\limits_{t-r_i(t)}^{t} h_i'(x(u))y(u)du - e_3(t,x(t))\eta'(t) + m(t,x,y). \end{aligned} \tag{11}$$

From now on, we shall write $f(t,x,y),\ g(t,x,y),\ m(t,x,y)$ as $f(.),\ g(.),\ m(.)$ respectively for brevity.

**Basic assumptions.** Let $\xi,\ \zeta,\ a_i,\ b_i,\ d_i,\ \beta_i,\ \lambda,\ \kappa,\ \kappa_1,\ldots,\kappa_7,\ \alpha_i,\ F_i$, and $\Delta_0$ be positive constants such that:

(i) $b_i \le k_i(t) \le a_i,\ 0 < k_i'(t) \le \beta_i,\ 0 \le r_i(t) \le \lambda,\ 0 \le r_i'(t) \le \kappa_1 \in (0,1)$;

(ii) $\kappa_2 \le e_1(t) \le \kappa_3,\ \kappa_4 \le e_2(t) \le \kappa_5 < 1$;

(iii) $\kappa_6 \le f(.) \le \zeta,\ \kappa_7 \le \dfrac{g(.)}{x} \le \xi$;

(iv) $h_i(0) = 0,\ \alpha_i \le \dfrac{h_i(x)}{x} \le F_i,\ x \ne 0,\ 0 < h_i'(x) = \dfrac{dh_i(x)}{dx} \le d_i$;

(v) $0 < \dfrac{e_3(t,x)}{x} \le \kappa,\ x \ne 0$;

(vi) $|m(.)| \le \Delta_0$.

**Theorem 3.1.** *Further to assumptions (i)–(v) listed above, we have*

$$\lambda < \frac{1}{\sum_{i=1}^{n} d_i a_i} \min\left\{\frac{M_2}{\xi\zeta}, \frac{M_3}{M_1}\right\}, \tag{12}$$

*where*

$$M = \xi - \left[\zeta^2(\kappa_3 + \kappa_5) + \kappa_5\xi\right] > 0,$$

$$M_1 = \frac{(\zeta^2 + \xi)(2 - \kappa_1) + \zeta\xi}{(1 - \kappa_1)} > 0,$$

$$M_2 = 2\zeta\xi\left(\sum_{i=1}^{n} b_i\alpha_i + \kappa_4\kappa_7\right) - 2\left(\zeta^2 + \xi\right)\left(\sum_{i=1}^{n} \beta_i F_i + \kappa^2\right) - M\xi > 0,$$

*and*

$$M_3 = 2\kappa_2\kappa_6\left(\zeta^2 + \xi\right) - \xi(2\zeta + M) > 0.$$

*Then the trivial solution of (11) when $m(.) \equiv 0$ is uniformly stochastically asymptotically stable.*

Proof of Theorem 3.1 shall be established by using the LKF $V(t) = V(t, x(t), y(t))$ given by

$$\begin{aligned} 2V(t) &= (\xi\, x + \zeta\, y)^2 + \xi\, y^2 + 2\left(\zeta^2 + \xi\right)\sum_{i=1}^{n} k_i(t) \int\limits_{0}^{x} h_i(\tau)\, d\tau \\ &\quad + \sum_{i=0}^{n} \gamma_i \int\limits_{-r_i(t)}^{0} \int\limits_{t+\ell}^{t} y^2(\Delta)\, d\Delta\, d\ell, \end{aligned} \tag{13}$$

where $\zeta,\ \xi$ are positive constants defined above and $\gamma_i > 0$ are constants to be determined shortly.

***Proof.*** If $x(t) = y(t) = 0,$ then $V(0) = 0.$ Following the conditions of the theorem, we get

$$2V(t) = (\xi\, x + \zeta\, y)^2 + \xi\, y^2 + 2\big(\zeta^2 + \xi\big) \sum_{i=1}^{n} k_i(t) \int_0^x h_i(\tau)\, d\tau$$

$$+ \sum_{i=0}^{n} \gamma_i \int_{-r_i(t)}^{0} \int_{t+\ell}^{t} y^2(\Delta)\, d\Delta\, d\ell$$

$$\geq \xi\, y^2 + \big(\zeta^2 + \xi\big) \sum_{i=1}^{n} b_i \alpha_i\, x^2 \geq C_1 \big\{x^2 + y^2\big\},$$

$$C_1 = \min\left\{\xi, \big(\zeta^2 + \xi\big) \sum_{i=1}^{n} b_i \alpha_i\right\}.$$

It is understood that the term with the double integrals in $V(t)$ is not negative.

Also, by using

$$2|uv| \leq u^2 + v^2, \tag{14}$$

we have

$$2V(t) \leq 2\xi^2\, x^2 + 2\zeta^2\, y^2 + \xi\, y^2 + \big(\zeta^2 + \xi\big) \sum_{i=1}^{n} a_i \digamma_i\, x^2$$

$$+ \sum_{i=0}^{n} \gamma_i \int_{-r_i(t)}^{0} \int_{t+\ell}^{t} y^2(\Delta)\, d\Delta\, d\ell \leq C_2 \big\{x^2 + y^2\big\} + \lambda \sum_{i=0}^{n} \int_{t-r_i(t)}^{t} y^2(\Delta)\, d\Delta,$$

where

$$C_2 = \max\left\{2\xi^2 + \big(\zeta^2 + \xi\big) \sum_{i=1}^{n} a_i \digamma_i,\ 2\zeta^2 + \zeta\right\}.$$

Therefore,

$$C_1 \big\{x^2 + y^2\big\} \leq V(t) \leq C_2 \big\{x^2 + y^2\big\} + \lambda \sum_{i=0}^{n} \int_{t-r_i(t)}^{t} y^2(\Delta)\, d\Delta. \tag{15}$$

From (15), we can conclude that for some positive constants $p_1$ and $p_2,$ the LKF $V(t)$ satisfies

$$\|x\|^{p_1} \leq V(t, x) \leq \|x\|^{p_2}. \tag{16}$$

We now use Itô formula to obtain the derivative of $V(t)$ along (11) as given below:

$$LV_S = -\zeta\, \xi\, e_2(t)\, \frac{g(.)}{x}\, x^2 - \big[\zeta^2\, e_1(t)\, f(.) - \xi(\zeta - e_1(t)\, f(.))\big] y^2 + \sum_{i=1}^{n} \gamma_i r_i(t) y^2$$

$$+\left(\zeta^2+\xi\right)e_3^2(t,x(t))-\sum_{i=1}^{n}\gamma_i(1-\dot r_i(t))\int\limits_{t-r_i(t)}^{t}y^2(u)\,du$$

$$+\left(\zeta^2+\xi\right)\sum_{i=1}^{n}k_i'(t)\int\limits_{0}^{x}h_i(\nu)\,d\nu+\sum_{k=1}^{3}\theta_k, \tag{17}$$

where

$$\theta_1=\left[\xi^2-\left(\xi\,\zeta\,e_1(t)f(.)+e_2(t)\,\frac{g(.)}{x}\left(\zeta^2+\xi\right)\right)\right]xy,$$

$$\theta_2=\left[\zeta\,\xi x+\left(\zeta^2+\xi\right)y\right]\sum_{i=1}^{n}k_i(t)\int\limits_{t-r_i(t)}^{t}h_i'(x(u))y(u)\,du,$$

$$\theta_3=-\zeta\,\xi\,x\sum_{i=1}^{n}k_i(t)h_i(x).$$

From the conditions of the theorem and inequality (14), we obtain

$$\theta_1\le\frac{1}{2}\,\xi M\left\{x^2+y^2\right\},$$

where

$$M=\xi-\left[\zeta^2(\kappa_3+\kappa_5)+\kappa_5\xi\right]>0\quad\text{and}\quad\kappa_5<1.$$

Also, from the conditions of the theorem and inequality (14), we get

$$\theta_2\le\frac{1}{2}\,\lambda\left[\zeta\,\xi\,x^2+\left(\zeta^2+\xi\right)y^2\right]\sum_{i=1}^{n}d_i\,a_i+\frac{1}{2}\left[\zeta^2+\zeta\,\xi+\xi\right]\sum_{i=1}^{n}d_ia_i\int\limits_{t-r_i(t)}^{t}y^2(s)\,ds.$$

Similarly, we have from the conditions of the theorem

$$\theta_3=-\zeta\xi x\sum_{i=1}^{n}k_i(t)\,\frac{h_i(x)}{x}\,x\le-\zeta\,\xi\sum_{i=1}^{n}b_i\alpha_ix^2.$$

Putting the estimates obtained for $\theta_i,\ i=1,2,3,$ into (17), we have

$$LV_S\le-\left[\zeta\,\xi\sum_{i=1}^{n}b_i\alpha_i+\zeta\xi e_2(t)\,\frac{g(.)}{x}-\frac{1}{2}\,M\xi\right.$$

$$\left.-\frac{1}{2}\,\lambda\zeta\xi\sum_{i=1}^{n}d_ia_i-\left(\zeta^2+\xi\right)\sum_{i=1}^{n}\beta_i F_i-\left(\zeta^2+\xi\right)\kappa^2\right]x^2$$

$$-\left[\zeta^2e_1(t)f(.)-\xi(\zeta-e_1(t)f(.))-\frac{1}{2}\,M\xi-\frac{1}{2}\,\lambda\left(\zeta^2+\xi\right)\sum_{i=1}^{n}d_ia_i-\sum_{i=1}^{n}\gamma_ir_i(t)\right]y^2$$

$$+\frac{1}{2}\left[\big(\zeta^2+\zeta\xi+\xi\big)\sum_{i=1}^{n}d_ia_i-2\sum_{i=1}^{n}\gamma_i(1-\kappa_1)\right]\int\limits_{t-r_i(t)}^{t}y^2(s)\,ds$$

$$\leq-\Bigg[\zeta\xi\sum_{i=1}^{n}b_i\alpha_i+\zeta\xi e_2(t)\,\frac{g(.)}{x}-\frac{1}{2}\,M\xi$$

$$-\frac{1}{2}\,\lambda\zeta\xi\sum_{i=1}^{n}d_ia_i-\big(\zeta^2+\xi\big)\sum_{i=1}^{n}\beta_i F_i-\big(\zeta^2+\xi\big)\kappa^2\Bigg]x^2$$

$$-\left[\zeta^2e_1(t)f(.)-\xi(\zeta-e_1(t)f(.))-\frac{1}{2}\,M\xi-\frac{1}{2}\,\lambda\big(\zeta^2+\xi\big)\sum_{i=1}^{n}d_ia_i-\sum_{i=1}^{n}\gamma_ir_i(t)\right]y^2$$

$$+\frac{1}{2}\sum_{i=1}^{n}\big[\big(\zeta^2+\zeta\xi+\xi\big)d_ia_i-2\gamma_i(1-\kappa_1)\big]\int\limits_{t-r_i(t)}^{t}y^2(s)\,ds.$$

Let us take

$$\gamma_i=\frac{\big(\zeta^2+\zeta\xi+\xi\big)d_ia_i}{2(1-\kappa_1)}\geq 0\quad\forall i=1,\ldots,n.$$

It then implies that

$$LV_S\leq-\Bigg[\zeta\xi\sum_{i=1}^{n}b_i\alpha_i+\zeta\xi e_2(t)\,\frac{g(.)}{x}-\frac{1}{2}\,M\xi$$

$$-\frac{1}{2}\,\lambda\zeta\xi\sum_{i=1}^{n}d_ia_i-\big(\zeta^2+\xi\big)\sum_{i=1}^{n}\beta_i F_i-\big(\zeta^2+\xi\big)\kappa^2\Bigg]x^2$$

$$-\Bigg[\zeta^2e_1(t)f(.)-\xi(\zeta-e_1(t)f(.))-\frac{1}{2}\,M\xi$$

$$-\frac{1}{2}\,\lambda\big(\zeta^2+\xi\big)\sum_{i=1}^{n}d_ia_i-\frac{\big(\zeta^2+\zeta\xi+\xi\big)}{2(1-\kappa_1)}\,\lambda\sum_{i=1}^{n}d_ia_i\Bigg]y^2.$$

Let us take $M_1=\dfrac{\big(\zeta^2+\xi\big)(2-\kappa_1)+\zeta\xi}{(1-\kappa_1)}>0$. Then we have

$$LV_S\leq-\frac{1}{2}\Bigg[2\zeta\xi\sum_{i=1}^{n}b_i\alpha_i+2\zeta\xi e_2(t)\,\frac{g(.)}{x}$$

$$-2\big(\zeta^2+\xi\big)\left(\sum_{i=1}^{n}\beta_i F_i+\kappa^2\right)-M\xi-\lambda\zeta\xi\sum_{i=1}^{n}d_ia_i\Bigg]x^2$$

$$-\frac{1}{2}\left[2\zeta^2e_1(t)f(.)-2\xi(\zeta-e_1(t)f(.))-M\xi-\lambda M_1\sum_{i=1}^{n}d_ia_i\right]y^2$$

$$\leq -\frac{1}{2}\left[2\zeta\xi\left(\sum_{i=1}^{n} b_i\,\alpha_i + \kappa_4\kappa_7\right) - 2\big(\zeta^2+\xi\big)\left(\sum_{i=1}^{n}\beta_i F_i + \kappa^2\right) - M\xi - \lambda\zeta\xi\sum_{i=1}^{n} d_i a_i\right]x^2$$

$$-\frac{1}{2}\left[2\kappa_2\kappa_6\big(\zeta^2+\xi\big) - 2\xi\zeta - M\xi - \lambda M_1\sum_{i=1}^{n} d_i a_i\right]y^2$$

$$\leq -\frac{1}{2}\left[M_2 - \lambda\zeta\xi\sum_{i=1}^{n} d_i a_i\right]x^2 - \frac{1}{2}\left[M_3 - \lambda M_1\sum_{i=1}^{n} d_i a_i\right]y^2,$$

where

$$M_2 = 2\zeta\xi\left(\sum_{i=1}^{n} b_i\alpha_i + \kappa_4\kappa_7\right) - 2\big(\zeta^2+\xi\big)\left(\sum_{i=1}^{n}\beta_i F_i + \kappa^2\right) - M\xi > 0$$

and

$$M_3 = 2\kappa_2\kappa_6\big(\zeta^2+\xi\big) - \xi(2\zeta + M) > 0.$$

Then

$$LV_S \leq -M_4\big\{x^2+y^2\big\}, \tag{18}$$

provided $\lambda$ satisfied (12) and $M_4$ is a positive constant.

Thus, from (15) and (18) we establish Theorem 3.1.

**Theorem 3.2.** *Assuming that assumptions (i) – (vi) of Basic assumptions above hold. Then solutions of (11) are uniformly stochastically bounded and provided inequality (12) hold.*

***Proof.*** We still rely on the LKF (13) to provide the proof. It has been shown earlier that (13) satisfied (15). Hence, we can go ahead to obtain the derivative using Ito formula when $m(.) \neq 0$:

$$LV_B = LV_S + \big[\zeta\xi x + \big(\zeta^2+\xi\big)y\big]m(.), \tag{19}$$

$LV_S$ is as defined in (17). From the assumption (vi) under Basic assumptions, we have

$$\big[\zeta\xi x + \big(\zeta^2+\xi\big)y\big]m(.) \leq \big|\zeta\xi x + \big(\zeta^2+\xi\big)y\big|\,|m(.)| \leq C_3(|x|+|y|),$$

where $C_3 = \Delta_0\max\big\{\zeta\xi, \zeta^2+\xi\big\}$. Therefore,

$$\begin{aligned} LV_B &\leq -M_4\big\{x^2+y^2\big\} + C_3(|x|+|y|) \\ &= -\frac{M_4}{2}\big\{x^2+y^2\big\} - \frac{M_4}{2}\Big(\big(|x| - M_4^{-1}C_3\big)^2 + \big(|y| - M_4^{-1}C_3\big)^2\Big) + M_4^{-1}C_3^2 \\ &\leq -\frac{M_4}{2}\left\{x^2+y^2\right\} + C_4, \end{aligned} \tag{20}$$

where $C_4 = M_4^{-1}C_3^2$. By inequality (15), condition (i) of Lemma 2.10 is satisfied with $j = p_1 = 2$ and $k = p_2 = 2$. Also, the condition (ii) of Lemma 2.10 is satisfied by taking $\alpha(t) = \dfrac{M_4}{2}$, $\psi(t) = C_4$, and $\overline{\xi} = 2$ in (20). Similarly, condition (iii) of Lemma 2.20 is realized with $\overline{\xi} = k = 2$ and $\mu = 0$. Furthermore, we show that inequality (10) of Corollary 2.11 holds as follows:

$$\int_{t_0}^{t}(\mu\alpha(u) + \psi(u))e^{-\int_u^t \alpha(\varepsilon)d\varepsilon}\,du = C_4\int_{t_0}^{t} e^{-\frac{M_4}{2}\int_u^t d\varepsilon}\,du \leq \frac{2C_4}{M_4} = \frac{2C_3^2}{M_4^2} \quad \forall t \geq t_0 \geq 0. \tag{21}$$

Thus, inequality (10) is realized from inequality (21) with $B = \dfrac{2C_3^2}{M_4^2} > 0$.

Going further, we establish inequality (7) of Assumption A2.2 by using (13) as in

$$\begin{aligned}\big|\Theta_{x_i}(t,x(t))\Omega_{ik}(t,x(t))\big| &= |V_{x_i}(t,x(t))\Omega_{ik}(t,x(t))|\\ &\le \kappa|x|\big(\zeta\xi|x| + \big(\xi+\zeta^2\big)|y|\big)\\ &\le \frac{1}{2}\,\kappa\big(\big(2\xi\zeta+\xi+\zeta^2\big)x^2 + \big(\xi+\zeta^2\big)y^2\big)\\ &\le C_5\big\{x^2+y^2\big\} := p(t), \end{aligned}\tag{22}$$

where $C_5 = \dfrac{1}{2}\,\kappa\big(2\xi\zeta+\xi+\zeta^2\big)$. We also establish inequality (8) of Assumption 2.8 using (22) as follows:

$$\int_{t_0}^{T} p^2(t)\,dt = C_5^2\int_{t_0}^{T}\big(x^2(t)+y^2(t)\big)^2\,dt < \infty \tag{23}$$

for any specified $0 \le t_0 \le T < \infty$.

Finally, we conclude the proof of the theorem by showing inequality (9) of Lemma 2.9 in what follows.

$$\begin{aligned} E^{x_0}\|x(t,x_0)\| &\le \left[V(t_0,x_0)e^{-\int_{t_0}^{t}\alpha(\varepsilon)d\varepsilon} + \int_{t_0}^{t}(\mu\alpha(u)+\psi(u))e^{-\int_{u}^{t}\alpha(\varepsilon)d\varepsilon}\,du\right]^{1/j}\\ &= \left[C_6\big(x^2(0),y^2(0)\big) + \frac{2C_4}{M_4}\right]^{1/2} \quad \forall t\ge t_0\ge 0, \end{aligned}\tag{24}$$

where $C_6 = \exp\!\left\{-\dfrac{M_4}{2}\,(t-t_0)\right\}$.

Since all the conditions of Lemma 2.10 and Corollary 2.11 have been satisfied, then the solutions of (11) are uniformly stochastically bounded.

**4. Example.** We give two examples which are particular cases of (11) when $n=1$.

***Example 4.1*** (stability)**.** Consider the following particular case of (25) with $m(.)=0$:

$$\begin{aligned} &x'' + \big(2+0.1e^{-t}\big)\left(\frac{4}{5} + \frac{1}{t^2+5+x^2+(x')^2}\right)x'\\ &\quad + \left(\frac{1}{3} - \frac{1}{6+t^2}\right)\left(4x + \frac{2x}{2+t^2+\sin xx'}\right)\\ &\quad + \left(3 - \frac{1}{1+t}\right)(3(x(t-r_1(t))) + \sin x(t-r_1(t)))\\ &\quad + \left(x - \frac{x}{1+e^{t^2}+x^2}\right)\eta'(t) = 0. \end{aligned}\tag{25}$$

On letting $x' = y$ in (25) we have

$$
\begin{aligned}
x' &= y,\\
y' &= -\left(2+0.1e^{-t}\right)\left(\frac{4}{5}+\frac{1}{t^2+5+x^2+y^2}\right)y-\left(\frac{1}{3}-\frac{1}{6+t^2}\right)\left(4x+\frac{2x}{2+t^2+\sin xy}\right)\\
&\quad-\left(3-\frac{1}{1+t}\right)(3x+\sin x)+\left(3-\frac{1}{1+t}\right)\int\limits_{t-r_1(t)}^{t}(3+\cos x(u))y(u)\,du \qquad (26)\\
&\quad-\left(x-\frac{x}{1+e^{t^2}+x^2}\right)\eta'(t).
\end{aligned}
$$

From (26), we have these estimates:

(i) $2 \le e_1(t) = 2+0.1e^{-t} \le 2.1$ and $\frac{1}{6} \le e_2(t) = \frac{1}{3}-\frac{1}{6+t^2} \le \frac{1}{3}$,

(ii) $e_3(t,x) = x-\frac{x}{1+e^{t^2}+x^2}$ and $0 \le \frac{e_3(t,x)}{x} = 1-\frac{1}{1+e^{t^2}+x^2} < 1$,

(iii) $h(x) = 3x+\sin x,\ 3 \le \frac{h(x)}{x} = 3+\frac{\sin x}{x} \le 4$, and $0 < h'(x) = 3+\cos x \le 4$,

(iv) $r(t) = \frac{0.01}{1+e^{-t}} \le 0.01$ and $r'(t) = \frac{e^{-t}}{(1+e^{-t})^2} \le 0.25$,

(v) $\frac{4}{5} \le f(.) = \frac{4}{5}+\frac{1}{5+t^2+x^2+y^2} \le 1$,

(vi) $g(.) = 4x+\frac{2x}{2+t^2+\sin xy}$ and $4 \le \frac{g(.)}{x} = 4+\frac{2}{2+t^2+\sin xy} \le 5$,

(vii) $2 \le k_1(t) = 3-\frac{1}{1+t} \le 3$ and $k_1'(t) = \frac{1}{(1+t)^2} \le 1$.

Thus, from (i) – (vii) above we have $\kappa = 1,\ \kappa_1 = 0.25,\ \kappa_2 = 2,\ \kappa_3 = 2.1,\ \kappa_4 = \frac{1}{6},\ \kappa_5 = \frac{1}{3},$ $\kappa_6 = \frac{4}{5},\ \kappa_7 = 4,\ a_1 = 3,\ b_1 = 2,\ d_1 = 4,\ \beta_1 = 1,\ \alpha_1 = 3,\ \lambda = 0.01,\ \zeta = 1,\ \xi = 5,$ and $F_1 = 4$. Hence,

$$
M = \xi-\left[\zeta^2(\kappa_3+\kappa_5)+\kappa_5\xi\right] = 5-\left[\frac{7.3}{3}+\frac{5}{3}\right] = 5-4.1 = 0.9 > 0,
$$

$$
M_1 = \frac{\left(\zeta^2+\xi\right)(2-\kappa_1)+\zeta\xi}{(1-\kappa_1)} = \frac{(1+5)(2-0.25)+5}{1-0.25} = \frac{15.5}{0.75} = 20.6667 > 0,
$$

$$
\begin{aligned}
M_2 &= 2\zeta\xi\left(\sum_{i=1}^{n} b_i\alpha_i+\kappa_4\kappa_7\right)-2\left(\zeta^2+\xi\right)\left(\sum_{i=1}^{n}\beta_i F_i+\kappa^2\right)-M\xi\\
&= 2(5)\left(6+\frac{2}{3}\right)-2(6)(5)-0.9(5) = \frac{13}{6} > 0.
\end{aligned}
$$

Similarly,

$$
M_3 = 2\kappa_2\kappa_6\left(\zeta^2+\xi\right)-\xi(2\zeta+M) = 2(2)\left(\frac{4}{5}\right)(6)-5(2+0.9) = \frac{47}{10} > 0.
$$

Finally,

$$\lambda = 0.01 < \frac{1}{\sum_{i=1}^{n} d_i a_i} \min\left\{\frac{M_2}{\xi\zeta}, \frac{M_3}{M_1}\right\}$$
$$= \frac{1}{12} \min\left\{\frac{13}{30}, \frac{728.5}{7.5}\right\}$$
$$< \min\{0.036, 97.133\} = 0.036,$$
$$0.01 < 0.036.$$

Conditions of Theorem 3.1 are met by this example. Hence, the trivial solution of (4.1) is uniformly stochastically asymptotically stable.

***Example 4.2*** (boundedness)**.** In addition to Example (4.1), let

$$m(.) = \frac{2 + \sin x}{2 + (t + x + y)^2}.$$

Then

$$|m(.)| \leq \frac{3}{2} = \Delta_0.$$

Let us now estimate the values of the constants used in the proof of Theorem 3.2:

$$C_3 = \Delta_0 \max\{\zeta\xi, \zeta^2 + \xi\} = \frac{3}{2} \max\{5, 6\} = 9,$$
$$M_4 = \frac{1}{2} \min\{M_2 - \lambda\xi\zeta d_1 a_1, M_3 - \lambda M_1\}$$
$$= \frac{1}{2}\left\{\frac{13}{6} - 0.01(5)(12), 4.7 - 0.01\left(\frac{1550}{75}\right)\right\} = 0.783,$$
$$C_4 = M_4^{-1} C_3^2 = \frac{81}{0.783} = 103.45,$$
$$C_5 = \frac{1}{2}\kappa\big(2\xi\zeta + \xi + \zeta^2\big) = \frac{1}{2}(1)(2(5) + 6) = 8,$$
$$C_6 = \exp\left\{-\frac{M_4}{2}(t - t_0)\right\} = \exp\{-0.3915(t - t_0)\}.$$

Now, we show that all the conditions (18) – (24) in the proof of Theorem 3.2 are fulfilled by Example 4.2:

$$LV_B \leq -M_4\{x^2 + y^2\} + C_3(|x| + |y|) = -0.783\{x^2 + y^2\} + 9(|x| + |y|),$$
$$\int_{t_0}^{t} (\mu\alpha(u) + \psi(u)) e^{-\int_u^t \alpha(\varepsilon) d\varepsilon}\, du = 103.45 \int_{t_0}^{t} e^{-\frac{M_4}{2}\int_u^t d\varepsilon}\, du$$
$$\leq \frac{2C_4}{M_4} e^{-\frac{M_4}{2}} = \frac{162}{0.783^2} = 264.24 > 0 \quad \forall t \geq t_0,$$

$$\left|V_{x_i}(t,x(t))\Phi_{ik}(t,x(t))\right| \le C_5\left\{x^2+y^2\right\} = 8\left\{x^2+y^2\right\} := p(t),$$

$$\int_{t_0}^{T} p^2(t)\,dt = 64\int_{t_0}^{T}\left(x^2(t)+y^2(t)\right)^2 dt < \infty$$

for specified $0 \le t_0 \le T < \infty$.

Finally,

$$E^{x_0}\|x(t,x_0)\| \le \left[C_6\left(x^2(0),y^2(0)\right) + \frac{2C_4}{M_4}\right]^{1/2}$$
$$= \left[e^{-0.3915(t-t_0)+29.36}\left(x^2(0),y^2(0)\right) + 264.24\right]^{1/2}$$
$$\le \left[\left(x^2(0),y^2(0)\right) + 264.24\right]^{1/2} \quad \forall t \ge t_0 \ge 0.$$

Consequently, solutions of Example 4.2 are uniformly stochastically bounded.

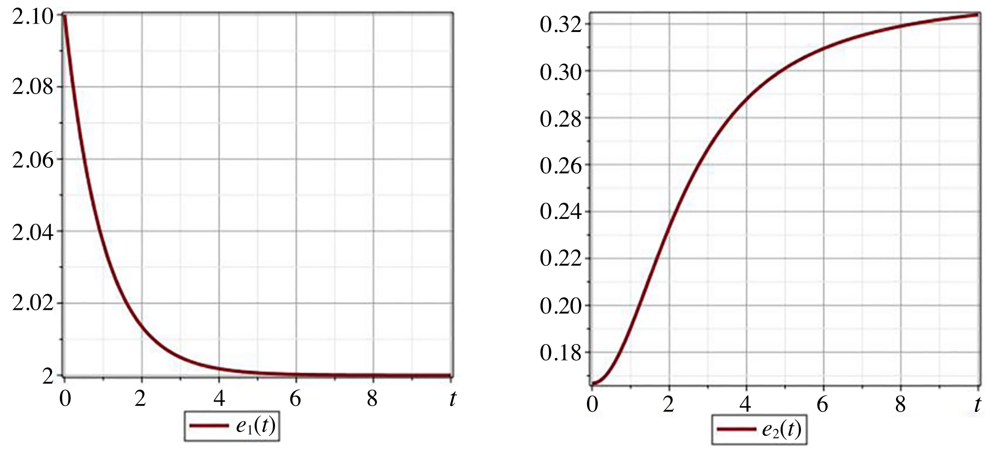


Fig. 1. Showing the behaviour of functions $e_1(t)$ and $e_2(t)$ when $t \in [0,10]$.

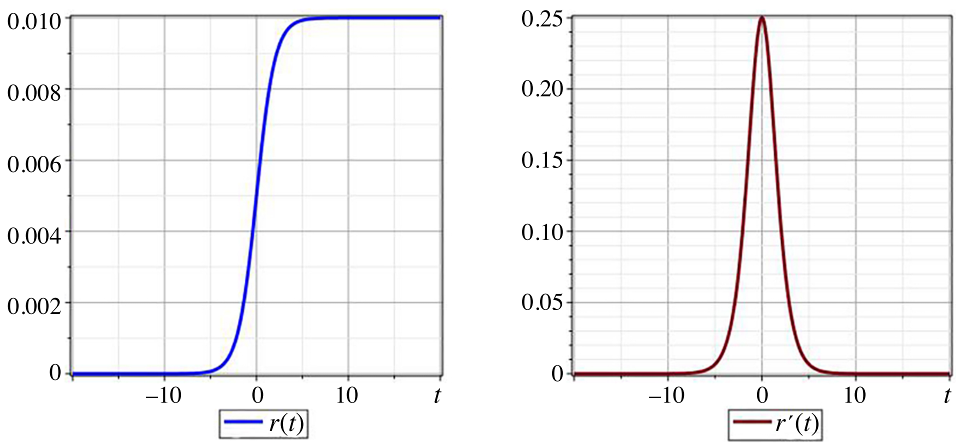


Fig. 2. The behaviour of functions $r(t)$ and $r'(t)$ when $t \in [-20,20]$.

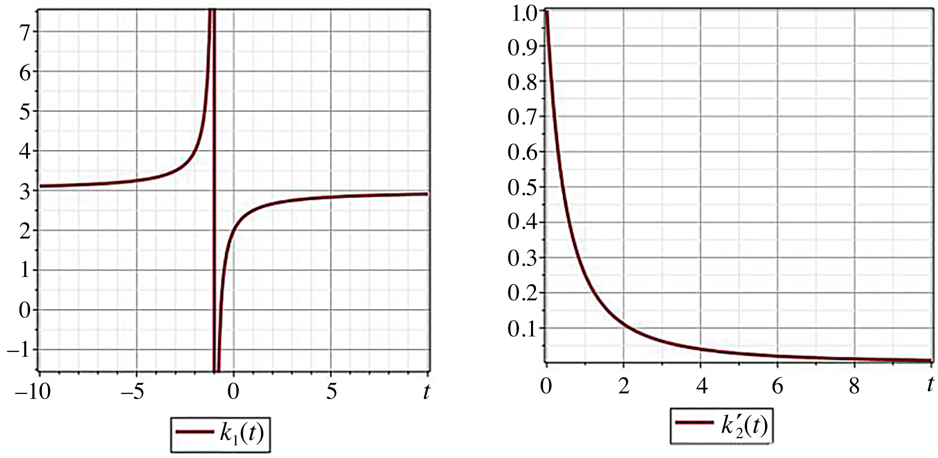


Fig. 3. The behaviour of functions $k_1(t)$ for $t \in [-10, 10]$ and $k_1'(t)$ for $t \in [0, 10]$.

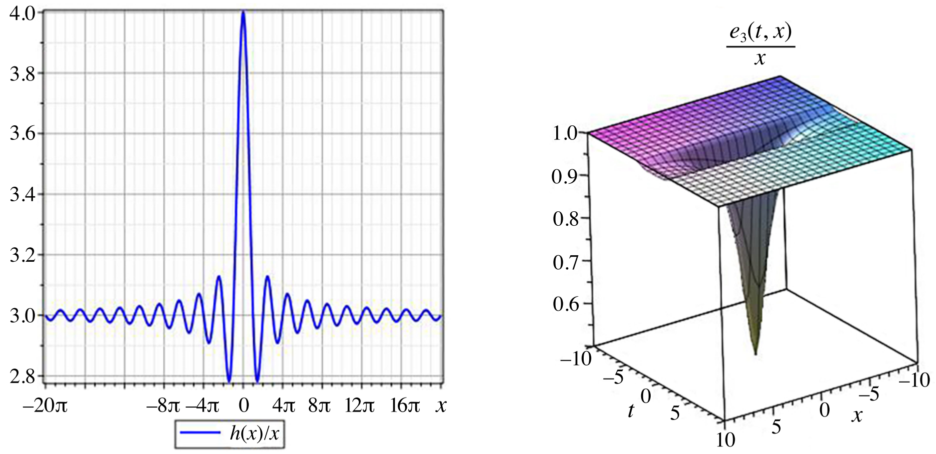


Fig. 4. The behaviour of functions $\dfrac{h(x)}{x}$ when $x \in [-20\pi, 20\pi]$ and $\dfrac{e_3(t,x)}{x}$ when $t, x \in [-10, 10]$.

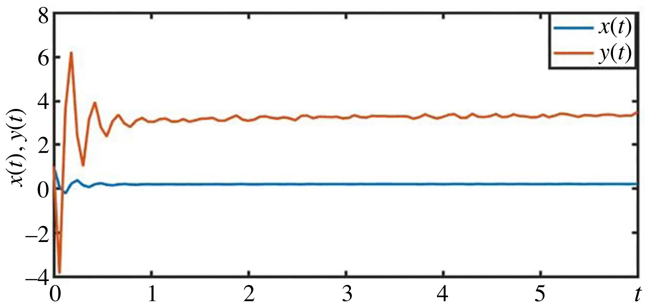


Fig. 5. Trajectories of solutions $x(t)$ and $y(t)$ for Example (4.2) when $t \in [0, 6]$.

**5. Conclusion.** In this study, a class of second order nonlinear SDE with multiple delay is considered. Necessary and sufficient assumptions for uniformly stochastically asymptotically stable of the trivial solution and UB of all solutions of the equation are given using a newly constructed Lyapunov functional. Examples are given to illustrate the credibility of our findings.

On behalf of all authors, the corresponding author states that there is no conflict of interest. All necessary data are included into the paper. All authors contributed equally to this work. The authors declare no special funding of this work.

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*Received 16.01.25*