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QUALITATIVE ANALYSIS OF A MATHEMATICAL MODEL OF TRANSMISSION AND CONTROL OF TUBERCULOSIS DISEASE

ЯКІСНИЙ АНАЛІЗ МАТЕМАТИЧНОЇ МОДЕЛІ ПЕРЕДАЧІ ТА КОНТРОЛЮ ЗАХВОРЮВАННЯ НА ТУБЕРКУЛЬОЗ

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This study presents a qualitative analysis of Susceptible – Latent – Infective-Recovered (SLIR) model for Tuberculosis (TB) transmission. The model captures the interaction between susceptible individuals, latent infection, active cases, and control measures. The model's behavior is analyzed through feasibility, positivity, equilibria, and stability. It is shown that TB infection can be stable for both endemic and disease-free equilibrium. The findings provide insight into TB transmission and control, informing early case detection, effective treatment, and evidence-based decision-making to reduce the global burden of TB.

Наведено якісний аналіз моделі передачі туберкульозу "сприйнятливий – латентно-інфікований – одужалий". Модель відображає взаємодію між сприйнятливими особами, латентною інфекцією, активними випадками та заходами контролю. Поведінку моделі проаналізовано з точки зору реалістичності, додатності, рівноваг і стійкості. Показано, що туберкульозна інфекція може бути стійкою як для ендемічної, так і вільної від хвороби рівноваги. Отримані результати дають уявлення про передачу туберкульозу та боротьбу з ним, сприяють ранньому виявленню випадків захворювання, ефективному лікуванню та прийняттю доказових рішень для зменшення глобального тягаря туберкульозу.

Introduction. Tuberculosis (TB) is an ancient infectious disease that has been documented as far back as 3000 BC, with evidence found in Neolithic, pre-Columbian, and early Egyptian remains. During the 17th and 18th centuries, TB was responsible for a quarter of all adult deaths in Europe [1]. It is caused by the bacillus Mycobacterium tuberculosis and primarily affects adults [2]. The spread of TB is driven by factors such as poverty, poor housing, overcrowding, malnutrition, and infections like Human Immunodeficiency Virus (HIV), as well as other conditions that weaken the immune system, such as malignancy, immunosuppressive therapy, chronic kidney disease, and diabetes [1].

According to Dr. Bertrand Odume in October 2022, Nigeria has the highest number of TB patients in Africa and ranks sixth globally [3]. TB is a family disease, so it is important to consider the antecedents, family life, environment, and illnesses of each family member. In 2021 and 2022, Aduh et al. [3] and James et al. [4], respectively, reported that Nigeria is ranked as the top country in Africa and sixth in the world for TB.

Several studies have focused on the diagnosis and risk factors of TB. In 2014, Rifat et al. [5] examined the disease burden and attributable risk factors of TB in Nigeria from 1990 to 2013. They concluded that "Multidrug-Resistant (MDR) TB (MDR-TB)" was found to be mostly associated with prior tuberculosis therapy. In 2023, Oladimeji et al. [6] studied the gender and drug-resistant (DR) TB in Nigeria. They characterized the gender distribution of DR-TB cases and the association between demographics and clinical data, such as age, treatment category, number of previous TB treatment cycles, and geopolitical zone, with gender. They found that the older male population from southsouthern Nigeria and women of childbearing age had lower incidence of DR-TB than men of the same age. In 2018, Gao and Huang [7] found that TB is a preventable and treatable infection. They extend the model of Liu and Zhang [8] by incorporating three control terms and applying the optimal control theory to the resulting model. They observed that implementation of three controls is most effective and less expensive among all the strategies. Thus, they concluded that in order to reduce tuberculosis threat all the three controls must be taken into consideration concurrently. Ugwu et al. [9] in 2021 presented findings on the prevalence of TB, DR HIV/TB, and TB co-infection in Enugu, Nigeria. They concluded that Enugu North geographical zone, Nigeria, has a high prevalence of both HIV and TB, including resistant TB and there is need to increase monitoring of individuals resident in this region. In 2012, Andrews et al. [10] reviewed cohort studies to assess the incidence of TB in individuals with latent TB and those without latent TB. In 2012, Belay et al. [11] conducted a cross-sectional study on diagnostic and treatment delay among TB patients in Afar Region, Ethiopia. They found that there is a long delay in the diagnosis and initiation of treatment and this was mainly attributable to the health system. In 2016, Al-Darraji et al. [12] investigated the prevalence of undiagnosed active TB among inmates in Malaysia. They concluded that the high prevalence of previously undiagnosed active TB in this prison highlights the inadequate performance of internationally recommended case-finding strategies and suggests that passive case-finding policies should be abandoned, especially in prison settings where HIV infection is prevalent. In 2014, Asefa and Teshome [13] conducted a study to determine the delay in initiating therapy for testing-positive pulmonary TB patients. They concluded that total delay in treatment of TB is still high in the study area. Patient's sex, perceived stigma, educational status and family size are significantly contributing factor for total delay.

In 2018, Blackwood and Childs [14] developed a disease model for individuals new to SIR-type models. Egonmwan and Okuonghae [15] also presented a mathematical model to study the impact of diagnosing and treating latent TB infections and active cases on disease transmission in 2019. They concluded that if treatment is made available for both latent and active cases of tuberculosis, then a decrease in the population's tuberculosis burden can be achieved by increasing the percentage of latent cases that are diagnosed and treated (even if only a small portion of active cases receive treatment the right away). In earlier studies in 2011, Okuonghae and Omosigho [16] conducted a qualitative and quantitative study of a mathematical model of TB in Benin City, Nigeria and concluded that it has been demonstrated that serious focus on tuberculosis awareness campaigns and the identification of active cough as a marker for TB infection can greatly lower the value of the reproduction number, hence lessening the severity of the disease in the presence of therapy. In a more recent study, Okuonghae [17] investigated qualitative features of a stochastic dynamic model of TB using case detection. In order to ensure the persistence of the disease in the presence of case detection, they constructed adequate requirements for the existence (and uniqueness) of an ergodic stationary distribution of the positive solutions of the model. In 2021, Akingbade and Ogundare [18] studied the boundedness and stability properties of solutions to a mathematical model of measles. They concluded that standards and prerequisites that ensured the asymptotic stability of both endemic and disease-free equilibriums (DFEs) were set. Additionally, R_0, the fundamental reproductive number, was discovered. Nwogu [19] studied the determinants of tuberculosis and diabetes mellitus co-morbidity among newly diagnosed tuberculosis patients attending NTBLCP clinics in abia state. He aimed that to determine the prevalence of TB-Tuberculosis Diabetes Mellitus (DM) co-morbidity and to assess the factors related to the occurrence of DM in TB patients among inhabitants of Abia State, Nigeria should be considered.

The COVID-19 outbreak in 2020, which presented symptoms similar to TB such as fever, difficulty breathing, fatigue, headache, sore throat, and coughing, prompted this study. The study aims to explain the transmission dynamics of TB and the need for control measures in our society.

The objective of this study is to conduct a qualitative analysis of a mathematical model of TB transmission and control, with a focus on enhancing our understanding of the disease dynamics and evaluating the effectiveness of various control strategies.

At this juncture, we need to point out that this paper is not involving use of any data or computational experiment.

The rest of the paper is organized as follows: Section 2 will discuss the model formulation for studying the transmission of TB in an open population; in Section 3 the stability analysis of the model equilibria is presented while Section 4 has our conclusion and recommendations.

The Table 1 presents the variables and parameters.

2. Tuberculosis model formulation. The progression of tuberculosis inside the entire population $N(t)$ can be done by four differential conditions. These conditions are represented by four groups of different individuals:

– the letter S represents the susceptible individuals of the population who have never contacted tuberculosis but as of now immunized against it;

Table 1. Variables/parameters for tuberculosis model

Variables	Description
$S(t)$	Number of susceptible individuals at time t
$L(t)$	Number of latent individuals at time t, i.e., individuals who have been infected, but not infectious
$I(t)$	Number of infected individuals at time t
$R(t)$	Number of recovered individuals at time t
$N(t)$	Total population at time t
Parameters	**Description**
θ	Incoming individuals (new born babies) who are immunized through vaccination
δ	Natural death/mortality rate from each compartment
μ	Rate at which susceptible individuals become latent
z	Rate at which susceptible individuals recovers naturally
β	Rate at which death occurs due to Infection
γ	Rate at which infected individuals recovers
k	Rate at which patiently infected individuals becomes infected
Ψ	Rate at which recovered individual becomes susceptible
R_0	Reproductive/regenerative number
Λ	Bounded space in R for SLIR model

– the letter L represents the latent individuals of the population who have been latently infected but not fully infected;

– the letter I represents the infective people of the population who have been infected and are able to transmit the contamination;

– the letter R represents the recovered people of the population that have recovered from tuberculosis.

2.1. Tuberculosis model description and formulation. The model consists of four mutually exclusive compartments of $S(t),\ L(t),\ I(t),\ R(t)$ of a deterministic ordinary differential equation (ODE), in a mixed homogeneous population. The total population at any time (t), denoted by $N(t)$, is the sum of the individual population in each compartment. Thus, $N(t) = S(t) + L(t) + I(t) + R(t)$. The model maintains the basic intuition of basic compartmental model [20].

In this model, we assumed that new born babies at constant rate θ who are immunized through vaccination enters the susceptible class. The susceptible class increases due to new born babies at the rate θ which enters the susceptible class and individuals who showed good immunity and moved from recovery class to susceptible class at the rate ψ. The susceptible class decreases due to natural death δ and the individuals who moved from susceptible class to latent class at the rate μ. The latent class increases at the rate μ due to susceptible individuals who becomes latently infected. The latent class decreases due to natural death at the rate δ. The infectious class increases at the rate k due to latently infected individuals who became infectious. The infectious compartment decreases due to natural death δ, death due to contamination at the rate β and those who are treated moved to recovery compartment at the rate γ. The recovered class decreases at the rate of δ, due to natural death and also due to individual with good immunity who become susceptible at the rate of ψ.

2.2. Presumptions of the model. The model is based on the following presumptions:

(1) The population size is largely open.

(2) The reality that the disease does not give immunity, as a result, treated and recovered individuals return to the susceptible class at a certain rate.

(3) The heterogeneity of the population. In other words, individuals within the population can be partitioned into compartments based on their epidemiological status.

(4) That the path of entering into the population is by birth and the path of exit is by natural death or tuberculosis related causes, i.e., no immigrants/emigrants.

(5) Individuals' resistance is conferred on them by immunization and contains a set time restrain and expires with time.

(6) In a compartment, the population size is that the epidemic model is unique and differentiable with respect to time t. To put it another way, the model is created by calculating the changes in population of the compartment.

(7) The individuals in each compartment have natural death rate δ.

Taking into account the above considerations, the tuberculosis model can be described by a systematic flow chat diagram as shown in the Fig. 1.

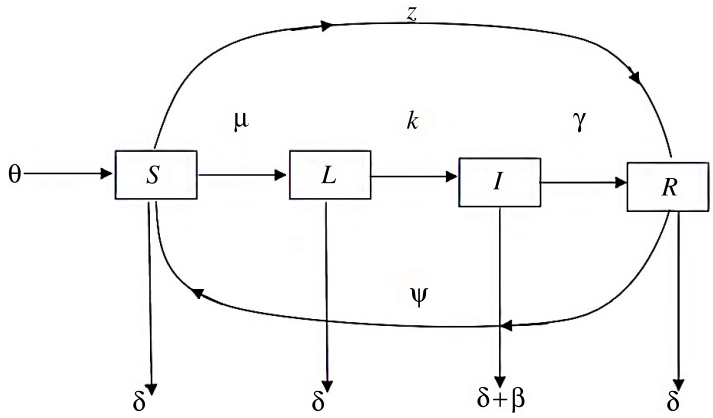

Fig. 1. A systematic diagram of a tuberculosis model.

2.3. Mathematical formulation of the model. Applying our above model description, formulation, assumptions and the systematic diagram, we have the following system of ordinary differential equations:

$$\begin{aligned}
\frac{dS}{dt} &= \theta + \Psi R - \mu SL - (\delta + z)S,\\
\frac{dL}{dt} &= \mu SL - (\delta + k)L,\\
\frac{dI}{dt} &= kL - (\delta + \beta)I - \gamma I,\\
\frac{dR}{dt} &= \gamma I + Sz - (\Psi + \delta)R.
\end{aligned} \tag{2.1}$$

2.4. Dynamics of the model. Let consider the entire population $N(t) = S(t) + L(t) + I(t) + R(t)$. Taking the derivative of $N(t)$, i.e., by adding up Equation (2.1), we get

$$\frac{dN}{dt} = \theta - \delta N(t) - \beta I. \tag{2.2}$$

This indicates that Equation (2.2) represent a change in population, known as population dynamics.

2.5. Basic properties of the model. We shall derive the basic properties of the model such as feasibility of the model and positivity of solutions.

2.5.1. Feasibility of the model. When we say feasibility of a model, we simply imply the region in which our solutions of the system of Equation (2.1) has a biological meaning.

Theorem 2.1. *Assume that Equation (2.2) holds and every solution (S, L, I, R) of the model within the system of equation (2.1) with initial condition in $\Re^4_+$, approaches to a member of a compact set (L) and remains within the compact set L in Λ as $t \to \infty$. Then the feasible solution which is a positively invariant set of the model is given by*

$$\Lambda = \left\{(S, L, I, R) \in \Re^4_+ : N \leq \frac{\theta}{\delta}\right\}. \tag{2.3}$$

Proof. From Equation (2.1), where changes of N leads to change all variables within the population (i.e., $N = S + L + I + R$), we recall Equation (2.2) as

$$\frac{dN}{dt} = \theta - \delta N(t) - \beta I.$$

Note: In the proof of Theorem (2.3), Equation (2.4) holds if either of I and β is zero:

$$\frac{dN}{dt} = \theta - \delta N(t). \tag{2.4}$$

Applying Birkhoff and Rota's theorem on the differential inequality in (2.4), we introduce the integrating factor (IF) which gives us

$$N(t) \leq \frac{\theta}{\delta} + ce^{-\delta t}. \tag{2.5}$$

Introducing the initial condition $t = 0$, we have

$$N(0) \leq \frac{\theta}{\delta} + c,$$

$$N(0) - \frac{\theta}{\delta} \leq c.$$

By substituting c in Equation (2.5), Equation (2.6) holds

$$\begin{aligned} N(t) &\leq \frac{\theta}{\delta} + e^{-\delta t}\left(N(0) - \frac{\theta}{\delta}\right) = \frac{\theta}{\delta} + e^{-\delta t}N(0) - e^{-\delta t}\,\frac{\theta}{\delta}, \\ N(t) &\leq e^{-\delta t}N(0) + \frac{\theta}{\delta}\big(1 - e^{-\delta t}\big). \end{aligned} \tag{2.6}$$

Considering the limit as $t \to \infty$, we have

$$N(t) \leq \frac{\theta}{\delta}.$$

These implies that $0 \leq N \leq \frac{\theta}{\delta}$.

Hence, we say the model in (2.1) is bounded within the region Λ.

The theorem is proved.

So the feasible solution is stated as

$$\Lambda = \left\{ (S, L, I, R) \in \Re_+^4 : N(t) \leq \frac{\theta}{\delta} \right\}$$

is a compact forward invariant set for the system of equations given by Equation (2.1). This shows that Λ is positively invariant. The solution of the system of equations (2.1) remains in Λ for all $t > 0$ and in this way the model has a biological meaning and epidemiologically well postured within the space Λ.

2.5.2. Positivity solution. The positivity of solution shows the property of nonnegativity of the solutions to our model Equation (2.1). For our model equation to have epidemiologically meaning, it is necessary to show that all its variables state are nonnegative for all time t. Let consider the Lemma 2.5.2 below.

Lemma 2.1. *Let the initial value of the system in Equation (2.1) be*

$$\{S(0), L(0), I(0), R(0) \geq 0 \in \Lambda\}.$$

Then the solution set $\{S(t), L(t), I(t), R(t)\}$ *of Equation (2.1) is positive for all* $t > 0$.

Proof. Consider the first equation in (2.1), it is expected that

$$\frac{dS}{dt} = \theta + \Psi R - \mu SL - (\delta + z)S \geq -(\delta + z)S \quad \text{for} \quad \mu \in [0, 1) \quad \text{and} \quad \mu \leq \frac{\theta}{SL},$$

$$\frac{dS}{dt} \geq -(\delta + z)S.$$

Integrating inequality (3.15) by separating variables gives

$$S(t) \geq S(0)e^{-(\delta+z)t}, \quad \text{since} \quad (\delta + z) > 0.$$

Similarly, the solutions of second, third, and fourth equations in the system of Equation (2.1) are obtained as

$$\frac{dL}{dt} = \mu SL - (\delta + k)L \geq -(\delta + k)L. \tag{2.7}$$

The solution of the inequality (2.7) is

$$L(t) \geq L(0)e^{-(\delta+k)t}, \quad \text{since} \quad (\delta + k) > 0, \tag{2.8}$$

$$\frac{dI}{dt} = kL - (\delta + \beta)I - \gamma I \geq -(\delta + \beta + \gamma)I. \tag{2.9}$$

The solution of the inequality (2.9) is

$$I(t) \geq I(0)e^{-(\delta+\beta+\gamma)t}, \quad \text{since} \quad (\delta+\beta+\gamma) > 0, \tag{2.10}$$

$$\frac{dR}{dt} = \gamma I + Sz - (\Psi+\delta)R \geq (\Psi+\delta)R.$$

The solution of the inequality (2.10) is

$$R(t) \geq R(0)e^{-(\Psi+\delta)t}, \quad \text{since} \quad (\Psi+\delta) > 0. \tag{2.11}$$

The results of the inequalities in Equations (2.7), (2.8), (2.10), and (2.11) show that the variables $S(t),\ L(t),\ I(t)$, and $R(t)$ are positive for all $t > 0$.

3. Stability analysis of the model equilibria. In this section, we shall introduce the model equilibria states and analyze these for stability. Then, the basic reproductive number (R_0) which tells the limit amount for thorough investigation of the asymptotic stability of the equilibria states and the prediction value required for removal of the disease will be obtained.

3.1. Equilibrium solutions. Let $E = (S, L, I, R) \in \Lambda$ be the equilibrium state of the system that is introduced in the system of equation (2.1). By obtaining the equilibrium states using these condition

$$\frac{dS}{dt} = \frac{dL}{dt} = \frac{dI}{dt} = \frac{dR}{dt} = 0.$$

That is,

$$\begin{aligned}
&\theta + \Psi R - \mu SL - (\delta+z)S = 0,\\
&\mu SL - (\delta+k)L = 0,\\
&kL - (\delta+\beta)I - \gamma I = 0,\\
&\gamma I + Sz - (\Psi+\delta)R = 0.
\end{aligned} \tag{3.1}$$

Let represent $\hat{\Lambda}$ and $\bar{\Lambda}$ in the boundary and interior of Λ in $\Re^4$ respectively. At these point by basic calculation, we can show that Equation (3.1) has two equilibria in $\Re^4_+$ the disease free equilibrium (DFE) $\hat{E}\big(\hat{S}, \hat{L}, \hat{I}, \hat{R}\big) \in \hat{\Lambda}$ and unique endemic equilibrium (EE) $\bar{E}\big(\bar{S}, \bar{L}, \bar{I}, \bar{R}\big) \in \bar{\Lambda}$.

3.2. DFE point. The DFE point of a model disease are its steady-state solution within the nonappearance of infection or disease. It is the state at which there's no disease within the population. At DFE, all the classes will be denoted with a hat. Let $\hat{E}\big(\hat{S}, \hat{L}, \hat{I}, \hat{R}\big)$ be the DFE state.

From model Equation (3.1), we have

$$\begin{aligned}
&\theta + \Psi\hat{R} - \mu\hat{S}\hat{L} - (\delta+z)\hat{S} = 0,\\
&\mu\hat{S}\hat{L} - (\delta+k)\hat{L} = 0,\\
&k\hat{L} - (\delta+\beta)\hat{I} - \gamma\hat{I} = 0,\\
&\gamma\hat{I} + \hat{S}z - (\Psi+\delta)\hat{R} = 0.
\end{aligned} \tag{3.2}$$

Substituting $\hat{I} = 0$ into Equation (3.2) gives

$$\begin{aligned} &\theta + \Psi\,\hat{R} - \mu\,\hat{S}\,\hat{L} - (\delta + z)\hat{S} = 0,\\ &\mu\,\hat{S}\hat{L} - (\delta + k)\hat{L} = 0,\\ &k\hat{L} = 0,\\ &\hat{S}z - (\Psi + \delta)\hat{R} = 0,\\ \hat{L} = 0, \qquad &\hat{S} = \frac{\theta(\Psi + \delta)}{\delta(\delta + z + \Psi)}, \qquad \hat{R} = \frac{z\theta}{\delta(\delta + z + \Psi)}. \end{aligned} \tag{3.3}$$

Hence, the DFE state of the model is

$$\hat{E} = \big(\hat{S}, \hat{L}, \hat{I}, \hat{R}\big) = \left(\frac{\theta(\Psi + \delta)}{\delta(\delta + z + \Psi)}, 0, 0, \frac{z\theta}{\delta(\delta + z + \Psi)}\right).$$

3.3. Local stability of the DFE point $\hat{E}$. Showing the stability or otherwise of the DFE state $\hat{E}$, we study the behaviour of the model population near the equilibrium solution. Hence, we decide the conditions that must be met for the DFE state to be stable and for the disease to be completely eradicated from the population. Recalling Equation (3.1), the system of equation reduces to

$$\begin{aligned} &\frac{dS}{dt} = \theta + \Psi R - \mu SL - (\delta + z)S = 0,\\ &\frac{dL}{dt} = \mu SL - (\delta + k)L = 0,\\ &\frac{dI}{dt} = kL - (\delta + \beta)I - \gamma I = 0,\\ &\frac{dR}{dt} = \gamma I + Sz - (\Psi + \delta)R = 0. \end{aligned} \tag{3.4}$$

To establish the stability of the equilibrium, the Jacobian matrix J of Equation (3.4) is computed and evaluated around the equilibrium state E.

Therefore, at DFE $\hat{E}$, the Jacobian matrix $\hat{J}$ is

$$\hat{J} = \begin{bmatrix} -\mu\hat{L} - \delta - z & -\mu\hat{S} & 0 & \psi\\ \mu\hat{L} & \mu\hat{S} - \delta - k & 0 & 0\\ 0 & k & -\delta - \beta - \gamma & 0\\ z & 0 & \gamma & -\psi - \delta \end{bmatrix}. \tag{3.5}$$

Substituting $\hat{S} = \dfrac{\theta(\Psi+\delta)}{\delta(\delta+z+\Psi)}$ and $\hat{L} = 0$ into Equation (3.5) gives

$$\hat{J} = \begin{bmatrix} -\delta - z & \dfrac{-\mu\theta(\Psi+\delta)}{\delta(\delta+z+\Psi)} & 0 & \psi \\ 0 & \dfrac{\mu\theta(\Psi+\delta)}{\delta(\delta+z+\Psi)} - \delta - k & 0 & 0 \\ 0 & k & -\delta-\beta-\gamma & 0 \\ z & 0 & \gamma & -\psi-\delta \end{bmatrix}. \tag{3.6}$$

The determinant of the matrix in Equation (3.6) is

$$\left|\hat{J} - I\lambda\right| = \begin{bmatrix} -\delta - z - \lambda & \dfrac{-\mu\theta(\Psi+\delta)}{\delta(\delta+z+\Psi)} & 0 & \psi \\ 0 & \dfrac{\mu\theta(\Psi+\delta)}{\delta(\delta+z+\Psi)} - \delta - k - \lambda & 0 & 0 \\ 0 & k & -\delta-\beta-\gamma-\lambda & 0 \\ z & 0 & \gamma & -\psi-\delta-\lambda \end{bmatrix}. \tag{3.7}$$

The solution of $\left|\hat{J} - I\lambda\right|$ in Equation (3.7) gives the eigenvalues as

$$\begin{aligned} \lambda_1 &= -\delta, \\ \lambda_2 &= -\Psi - \delta - z, \\ \lambda_3 &= \frac{\mu\theta\delta + \mu\theta\psi - \delta(\delta+z+\psi) - k\delta(\delta+z+\psi)}{\delta(\delta+z+\psi)}, \\ \therefore \lambda_3 &= \frac{\mu\theta\delta + \mu\theta\psi}{\delta(\delta+z+\psi)} - \delta - k, \\ \lambda_4 &= -\delta - \beta - \gamma. \end{aligned} \tag{3.8}$$

Lemma 3.1. *The DFE point $\hat{E}$ in Equation (2.1) is asymptotically stable if $\lambda_1, \lambda_2, \lambda_3, \lambda_4 < 0$ and unstable if at least one of $\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4$ is greater than zero for all $\theta,\ \Psi,\ \mu,\ \delta,\ z,\ k$, and γ are positive [18].*

Proof. The DFE point $\hat{E}$ is asymptotically stable if all the eigenvalues $\lambda_i,\ i = \overline{1,4}$, of $\hat{J}(\hat{E})$ satisfy Routh – Hurwitz criterion [21]: applying the Routh – Hurwitz criterion which state that "the number of roots of the characteristic equation with positive real parts (unstable) is equal to the number of changes of sign of the coefficients in the first column of the array" from Equation (3.8), we observed the first two eigenvalues $\lambda_1,\ \lambda_2$ and λ_4 has negative real parts. We build up the essential and adequate condition for the λ_3 to have negative real part in order for the DFE to be stable. From λ_3 we obtain

$$\frac{-[-\mu\theta(\delta+\psi) + (\delta+k)\delta(\delta+z+\psi)]}{\delta(\delta+z+\psi)} < 0.$$

By inequality (3.3) becomes

$$-\mu\theta(\delta+\psi)+(\delta+k)\delta(\delta+z+\psi)<0,$$

$$(\delta+k)\delta(\delta+z+\psi)>\mu\theta(\delta+\psi)$$

or

$$\mu\theta(\delta+\psi)<(\delta+k)\delta(\delta+z+\psi)\Rightarrow\mu\theta<\frac{(\delta+k)\delta(\delta+z+\psi)}{\delta+\psi}. \tag{3.9}$$

By using $\dfrac{\delta+\psi}{\delta(\delta+z+\psi)}$ to multiply Equation (3.9), we have

$$\Rightarrow\frac{\mu\theta(\delta+\psi)}{\delta(\delta+z+\psi)}<\delta+k. \tag{3.10}$$

The necessary and sufficient condition explains the inequality (3.10) for the DFE state $\hat{E}$ of the model to be asymptotically stable since it satisfies λ_3 to have a negative real part. Inequality (3.10) furtherly explain the total eradication of the infectious compartment which has a lower bound as $\dfrac{\mu\theta(\delta+\psi)}{\delta(\delta+z+\psi)}$.

3.4. Basic reproductive number (R_0) of the model. The basic reproductive number (R_0) is the mean number of secondary infections caused by a single infectious person with time. Heesterbeek and Dietz (1996) expressed that R_0 is one of the preeminent and most profitable thoughts that scientific considering has brought to plague hypothesis. We will utilize it to anticipate the series of infection and assess methodologies for controlling infection. To get it we use the largest eigenvalue λ_3 of our equilibrium state Jacobian matrix. This reproductive number R_0 is our threshold value or quantity. Recall that to establish the necessary and sufficient condition for λ_3 to have negative real part and for the DFE to be stable we set $\lambda_3<0$.

Let $\lambda_3<0$ then, we have

$$\frac{\mu\theta(\delta+\psi)}{\delta(\delta+z+\psi)}-(\delta+k)<0.$$

As we cross multiply

$$\frac{\mu\theta(\delta+\psi)}{\delta(\delta+z+\psi)}<(\delta+k).$$

Hence,

$$\frac{\mu\theta(\delta+\psi)}{\delta(\delta+z+\psi)(\delta+k)}<1. \tag{3.11}$$

Therefore, Equation (3.11) allows the definition of R_0 for the model as

$$R_0=\frac{\mu\theta(\delta+\psi)}{\delta(\delta+z+\psi)(\delta+k)}, \tag{3.12}$$

where $\delta(\delta+z+\psi)(\delta+k)\neq0$.

Remark 3.1. The threshold quantity R_0 defined in Equation (3.12) is the basic reproduction ratio of the infections for the nonlinear autonomous ordinary differential equations in (2.1).

Remark 3.2. Epidemiologically:

(i) if $R_0 < 1$, the event of the illness will diminish and the infection will in the long run be disposed of;

(ii) if $R_0 = 1$, the infection event will be steady;

(iii) if $R_0 > 1$, the event of the disease will increase. Contamination persevere.

Thus, we have establish the following.

Theorem 3.1. *The DFE $\hat{E}$ of the system in Equation (2.1) is locally asymptotically stable in η if $R_0 < 1$ and unstable if $R_0 > 1$ for Ψ, δ, z, k, γ, and θ are all positive.*

Proof. From Lemma 3.1, we see that $\lambda_1,\ \lambda_2,\ \lambda_4 < 0$. Then the DFE points $\hat{E}$ is locally asymptotically stable if $\lambda_3 < 0$.

By definition

$$R_0 = \frac{\mu\theta(\delta+\psi)}{\delta(\delta+z+\psi)(\delta+k)}.$$

By using the inequality in Equation (3.11) $R_0 < 1$, noting that $\lambda_3 < 0$ if and only if $R_0 < 1$. Therefore, DFE $\hat{E}$ of (2.1) is locally asymptotically stable. Otherwise, if $R_0 > 1,\ \lambda_3$ is positive. Thus, DFE point $\hat{E}$ of (2.1) is locally asymptotically unstable.

Theorem 3.5 is proved.

Remark 3.3. In view of Remark 3.2 to our model, we have:

(i) if $\mu\theta < \dfrac{(\delta+k)\delta(\delta+z+\psi)}{\delta+\psi}$, the event of the tuberculosis contamination will diminish and the illness will in the long run be disposed of, i.e., no epidemic;

(ii) if $\mu\theta = \dfrac{(\delta+k)\delta(\delta+z+\psi)}{\delta+\psi}$, the event of the tuberculosis contamination will be steady;

(iii) if $\mu\theta > \dfrac{(\delta+k)\delta(\delta+z+\psi)}{\delta+\psi}$ the event of the tuberculosis contamination will increase. Contamination continues, each person will produces more than one contaminations.

If the equilibrium $\hat{E}$ is globally asymptotically stable, at that point the disease will be eradicated from the population irrespective of the initial sizes of the four state variables.

3.5. Global stability of DFE. In this section, we prove the global stability of the DFE $\hat{E}$ when the basic reproductive number is less than or equal to unity.

Theorem 3.2. *The DFE $\hat{E}$, of the equilibrium (3.11) is globally asymptotically stable in Λ if $R_0 < 1$ [18].*

Proof. Since $R_0 < 1$, it follows that there exists $\epsilon_0 > 0$ such that

$$\mu\left(\frac{\theta(\delta+\psi)}{\delta(\delta+z+\psi)} + \epsilon_0\right) - (\delta+k) < 0.$$

Solving for the first equation in (3.3),

$$\frac{d\hat{S}(t)}{dt} \le \theta + \frac{\psi\theta z}{\delta(\delta+z+\psi)} - (\delta+z)\hat{S}(t) \Rightarrow \frac{d\hat{S}(t)}{dt} + (\delta+z)\hat{S}(t) \le \theta + \frac{\psi\theta z}{\delta(\delta+z+\psi)}.$$

Integrating factors $e^{\int(\delta+z)\,dt} \le e^{(\delta+z)t}$. Multiply through by the IF

$$e^{(\delta+z)t}\frac{d\hat{S}(t)}{dt} + e^{(\delta+z)t}(\delta+z)\hat{S}(t) \le \left(\theta + \frac{\psi\theta z}{\delta(\delta+z+\psi)}\right)e^{(\delta+z)t}.$$

Recalling product rule from left-hand side, we obtain

$$\hat{S}(t)e^{(\delta+z)t} \leq \left(\theta + \frac{\psi\theta z}{\delta(\delta+z+\psi)}\right)e^{(\delta+z)t}.$$

Integrate right-hand side, we have

$$\hat{S}(t)e^{(\delta+z)t} \leq \left(\theta + \frac{\psi\theta z}{\delta(\delta+z+\psi)}\right)\int e^{(\delta+z)t}\,dt,$$

$$\hat{S}(t)e^{(\delta+z)t} \leq \left(\frac{\theta\delta(\delta+z+\psi)+\psi\theta z}{\delta(\delta+z+\psi)}\right)\frac{e^{(\delta+z)t}}{\delta+z},$$

$$\hat{S}(t)e^{(\delta+z)t} \leq \left(\frac{\theta\delta(\delta+z+\psi)+\psi\theta z}{\delta(\delta+z+\psi)(\delta+z)}\right)e^{(\delta+z)t} + c.$$

Divide both side by IF, we get

$$\hat{S}(t) \leq \left(\frac{\theta\delta(\delta+z+\psi)+\psi\theta z}{\delta(\delta+z+\psi)(\delta+z)}\right) + ce^{(\delta+z)t}.$$

As $t \to \infty$

$$\hat{S}(t) \leq \frac{\theta(\delta^2+z\delta+\psi\delta+\psi z)}{\delta(\delta+z+\psi)(\delta+z)}.$$

By factorization

$$\hat{S}(t) \leq \frac{\theta(\delta(\delta+z)+\psi(\delta+z))}{\delta(\delta+z+\psi)(\delta+z)},$$

$$\hat{S}(t) \leq \frac{\theta((\delta+z)(\delta+\psi))}{\delta(\delta+z+\psi)(\delta+z)}.$$

Therefore,

$$\hat{S}(t) \leq \frac{\theta(\delta+\psi)}{\delta(\delta+z+\psi)}.$$

Considering the second Equation (3.3):

$$\frac{d\hat{L}(t)}{dt} \leq \mu\hat{S}\hat{L}(t) - (\delta+k)\hat{L}(t),$$

$$\frac{d\hat{L}(t)}{dt} = -\left((\delta+k) - \mu\left(\frac{\theta(\delta+\psi)}{\delta(\delta+z+\psi)} + \epsilon_0\right)\right)\hat{L}(t),$$

$$\frac{d\hat{L}(t)}{dt} + \left((\delta+k) - \mu\left(\frac{\theta(\delta+\psi)}{\delta(\delta+z+\psi)} + \epsilon_0\right)\right)\hat{L}(t) = 0. \tag{3.13}$$

By using the IF, we obtain

$$e^{\int\left((\delta+k)-\mu\left(\frac{\theta(\delta+\psi)}{\delta(\delta+z+\psi)}+\epsilon_0\right)\right)dt} = e^{\left((\delta+k)-\mu\left(\frac{\theta(\delta+\psi)}{\delta(\delta+z+\psi)}+\epsilon_0\right)\right)t}.$$

Multiply the IF by Equation (3.13)

$$e^{\left((\delta+k)-\mu\left(\frac{\theta(\delta+\psi)}{\delta(\delta+z+\psi)}+\epsilon_0\right)\right)t}\frac{d\hat{L}(t)}{dt}$$
$$+ e^{\left((\delta+k)-\mu\left(\frac{\theta(\delta+\psi)}{\delta(\delta+z+\psi)}+\epsilon_0\right)\right)t}\left((\delta+k)-\mu\left(\frac{\theta(\delta+\psi)}{\delta(\delta+z+\psi)}+\epsilon_0\right)\right)\hat{L}(t)=0,$$
$$\int\left[\frac{d}{dt}\,e^{\left((\delta+k)-\mu\left(\frac{\theta(\delta+\psi)}{\delta(\delta+z+\psi)}+\epsilon_0\right)\right)t}\hat{L}(t)\right]=0,$$
$$e^{\left((\delta+k)-\mu\left(\frac{\theta(\delta+\psi)}{\delta(\delta+z+\psi)}+\epsilon_0\right)\right)t}\hat{L}(t)=0.$$

Dividing both sides by IF and taking limit as $t\to\infty$, we have

$$\lim_{t\to\infty}\hat{L}(t)=0.$$

Considering the third Equation (3.3):

$$\frac{d\hat{R}(t)}{dt}=\frac{\theta z(\Psi+\delta)}{\delta(\delta+z+\Psi)}-(\psi+\delta)\hat{R}(t)\Rightarrow\frac{d\hat{R}(t)}{dt}+(\psi+\delta)\hat{R}(t)=\frac{\theta z(\Psi+\delta)}{\delta(\delta+z+\Psi)}.$$

By using IF, we get

$$e^{\int(\psi+\delta)\,dt}=e^{(\psi+\delta)t}.$$

Multiply sides by the IF:

$$\Rightarrow e^{(\psi+\delta)t}\,\frac{d\hat{R}(t)}{dt}+e^{(\psi+\delta)t}(\psi+\delta)\hat{R}(t)=e^{(\psi+\delta)t}\,\frac{\theta z(\Psi+\delta)}{\delta(\delta+z+\Psi)}.$$

Recalling the product rule from left-hand side and take integral of right-hand side,

$$e^{(\psi+\delta)t}\hat{R}(t)=\int e^{(\psi+\delta)t}\,\frac{\theta z(\Psi+\delta)}{\delta(\delta+z+\Psi)}\,dt,$$
$$e^{(\psi+\delta)t}\hat{R}(t)=e^{(\psi+\delta)t}\,\frac{\theta z(\Psi+\delta)}{\delta(\delta+z+\Psi)(\psi+\delta)}+c.$$

Divide both side by the IF:

$$\hat{R}(t)=\frac{\theta z}{\delta(\delta+z+\psi)}+c\,e^{-(\psi+\delta)t}.$$

As $t\to\infty$

$$\lim_{t\to\infty}\hat{R}(t)=\frac{\theta z}{\delta(\delta+z+\psi)}.$$

Therefore, the DFE is globally stable.

3.6. Existence and uniqueness of EE. The state in which the infectious disease is beyond eradication but is persistent in the population is the *Endemic Equilibrium state*. Consider the EE state $\bar{E} = (\bar{S}, \bar{L}, \bar{I}, \bar{R})$. Note that here none of the compartmental classes can be zero at equilibrium state. In order to obtain the EE state, one solves Equation (2.1):

$$\theta + \Psi\bar{R} - \mu\,\bar{S}\,\bar{L} - (\delta + z)\bar{S} = 0,$$

$$\mu\bar{S}\,\bar{L} - (\delta + k)\bar{L} = 0,$$

$$k\bar{L} - (\delta + \beta)\bar{I} - \gamma\,\bar{I} = 0,$$

$$\gamma\bar{I} + \bar{S}z - (\Psi + \delta)\bar{R} = 0.$$

The solution $\bar{E} = (\bar{S}, \bar{L}, \bar{I}, \bar{R}) \neq (0, 0, 0, 0)$, where

$$\begin{aligned}
\bar{S} &= \frac{\delta + k}{\mu},\\
\bar{L} &= -\frac{\big(\delta^3 + \delta^2 k + \delta^2\psi + \delta^2 z + \delta k\psi + \delta kz - \delta\mu\theta - \mu\psi\theta\big)(\beta + \delta + \gamma)}{\mu(\beta\delta^2 + \beta\delta k + \beta\delta\psi + \beta k\psi + \delta^3 + \delta^2\gamma + \delta^2 k + \delta^2\psi + \delta\gamma k + \delta\gamma\psi + \delta k\psi)},\\
\bar{I} &= -\frac{k\big(\delta^3 + \delta^2 k + \delta^2\psi + \delta^2 z + \delta k\psi + \delta kz - \delta\mu\theta - \mu\psi\theta\big)}{\mu(\beta\delta^2 + \beta\delta k + \beta\delta\psi + \beta k\psi + \delta^3 + \delta^2\gamma + \delta^2 k + \delta^2\psi + \delta\gamma k + \delta\gamma\psi + \delta k\psi)},\\
\bar{R} &= -\frac{-\beta\delta^2 z - 2\beta\delta kz - \beta k^2 z - \delta^3 z + \delta^2\gamma k - \delta^2\gamma z - 2\delta^2 kz + \delta\gamma k^2 - \delta\gamma kz - \delta k^2 z - \gamma k\mu\theta}{\mu(\beta\delta^2 + \beta\delta k + \beta\delta\psi + \beta k\psi + \delta^3 + \delta^2\gamma + \delta^2 k + \delta^2\psi + \delta\gamma k + \delta\gamma\psi + \delta k\psi)}.
\end{aligned} \tag{3.14}$$

The vector representation of solution in Equation (3.14) is

$$\begin{aligned}
&\bar{E}\big(\bar{S}, \bar{L}, \bar{I}, \bar{R}\big)\\
&= \left(\frac{\delta + k}{\mu}, -\frac{\big(\delta^3 + \delta^2 k + \delta^2\psi + \delta^2 z + \delta k\psi + \delta kz - \delta\mu\theta - \mu\psi\theta\big)(\beta + \delta + \gamma)}{\mu(\beta\delta^2 + \beta\delta k + \beta\delta\psi + \beta k\psi + \delta^3 + \delta^2\gamma + \delta^2 k + \delta^2\psi + \delta\gamma k + \delta\gamma\psi + \delta k\psi)},\right.\\
&\quad -\frac{k\big(\delta^3 + \delta^2 k + \delta^2\psi + \delta^2 z + \delta k\psi + \delta kz - \delta\mu\theta - \mu\psi\theta\big)}{\mu(\beta\delta^2 + \beta\delta k + \beta\delta\psi + \beta k\psi + \delta^3 + \delta^2\gamma + \delta^2 k + \delta^2\psi + \delta\gamma k + \delta\gamma\psi + \delta k\psi)},\\
&\quad \left. -\frac{-\beta\delta^2 z - 2\beta\delta kz - \beta k^2 z - \delta^3 z + \delta^2\gamma k - \delta^2\gamma z - 2\delta^2 kz + \delta\gamma k^2 - \delta\gamma kz - \delta k^2 z - \gamma k\mu\theta}{\mu(\beta\delta^2 + \beta\delta k + \beta\delta\psi + \beta k\psi + \delta^3 + \delta^2\gamma + \delta^2 k + \delta^2\psi + \delta\gamma k + \delta\gamma\psi + \delta k\psi)}\right).
\end{aligned}$$

Representing the EE state in term of reproductive number R_0, we obtain

$$\bar{E}\big(\bar{S},\bar{L},\bar{I},\bar{R}\big)$$

$$= \begin{cases} \bar{S} = \dfrac{\theta(\delta+\psi)}{R_0\delta(\delta+z+\psi)}, \\ \bar{L} = \dfrac{(R_0-1)(\psi+\delta)(\delta+\gamma+k)\theta}{R_0(\delta^3+\delta^2(2k+\psi+\gamma)\delta(\psi(2k+\gamma)+k(\gamma+k))+k^2\psi)}, \\ \bar{I} = \dfrac{(R_0-1)(\psi+\delta)k\theta}{R_0(\delta^3+\delta^2(2k+\psi+\gamma)\delta(\psi(2k+\gamma)+k(\gamma+k))+k^2\psi)}, \\ \bar{R} = \dfrac{\theta\big(\delta^3 z+(((R_0-1)\gamma+2z)k+z(\psi+\gamma))\delta^2}{\delta R_0(\delta^3+\delta^2(2k+\psi+\gamma)+\delta(k^2+(2\psi+\gamma)k+\psi\gamma)+k^2\psi)(\delta+z+\psi)} \\ \qquad +\dfrac{\theta\big(k^2 z+(((R_0-1)\psi+zR_0)\gamma+2z\psi)k+z\psi\gamma\big)\delta+k^2 z\psi\big)}{\delta R_0(\delta^3+\delta^2(2k+\psi+\gamma)+\delta(k^2+(2\psi+\gamma)k+\psi\gamma)+k^2\psi)(\delta+z+\psi)}. \end{cases}$$

It is evident from the above four equations that if $R_0<1$, then the model has no positive EE since $\bar{L}$, $\bar{I}$, and $\bar{R}$ will assume negative values which make no meaning biologically. Hence, to ensure the existence of a positive EE, we require $R_0>1$. Since $\bar{L},\bar{I},\bar{R}>0$ when $R_0>1$, the EE $\bar{E}$ is positive and $\bar{L},\bar{I},\bar{R}>0$. This is the condition for the existence and uniqueness of the EE for the EE for the system of Equation (2.1).

3.7. Global stability of EE.

Statement. *The EE $\bar{E}$ is globally asymptotically stable in Λ if $R_0>1$.*

Proof. Recalling that

$$R_0=\frac{\mu\theta(\delta+\psi)}{\delta(\delta+z+\psi)(\delta+k)}.$$

Considering the first equation, we can substitute the result of $\bar{L}$ and $\bar{R}$ in system (3.14)

$$\begin{aligned} &\frac{d\bar{S}}{dt}=\theta+\Psi\bar{R}-\mu\bar{L}\bar{S}(t)-(\delta+z)\bar{S}(t), \\ &\frac{d\bar{S}}{dt}+\big(\mu\bar{L}+\delta+z\big)\bar{S}(t)=\theta+\Psi\bar{R}. \end{aligned} \tag{3.15}$$

Solve for the IF:

$$\text{IF}=e^{\int\left(\mu\bar{L}+\delta+z\right)dt}=e^{(\mu\bar{L}+\delta+z)t}.$$

Multiplying both sides by IF

$$e^{(\mu\bar{L}+\delta+z)t}\,\frac{d\bar{S}}{dt}+e^{(\mu\bar{L}+\delta+z)t}\big(\mu\bar{L}+\delta+z\big)\bar{S}(t)=\big(\theta+\Psi\bar{R}\big)\,e^{(\mu\bar{L}+\delta+z)t}.$$

Recalling product rule $uv'+vu'=\dfrac{d}{dt}(uv)$

$$\frac{d}{dt}\Big[e^{(\mu\bar{L}+\delta+z)t}\,\bar{S}(t)\Big]=\big(\theta+\Psi\bar{R}\big)e^{(\mu\bar{L}+\delta+z)t}.$$

Integrating both sides

$$\int\frac{d}{dt}\Big[e^{(\mu\bar{L}+\delta+z)t}\bar{S}(t)\Big]=\big(\theta+\Psi\bar{R}\big)\int e^{(\mu\bar{L}+\delta+z)t}\,dt,$$

$$e^{(\mu\bar{L}+\delta+z)t}\,\bar{S}(t) = \frac{\big(\theta+\Psi\bar{R}\big)}{\big(\mu\,\bar{L}+\delta+z\big)}e^{(\mu\bar{L}+\delta+z)t}+c.$$

Divide both sides by the IF

$$\bar{S}(t) = \frac{\big(\theta+\Psi\,\bar{R}\big)}{\big(\mu\bar{L}+\delta+z\big)}+ce^{-(\mu\,\bar{L}+\delta+z)t}.$$

Taking limit as $t\to\infty$

$$\lim_{t\to\infty}\bar{S}(t) = \frac{\theta+\Psi\bar{R}}{\mu\bar{L}+\delta+z}. \tag{3.16}$$

Considering the third equation

$$\frac{d}{dt}\,\bar{I}(t) = k\bar{L}-(\delta+k)\bar{I}(t)-\gamma\,\bar{I}(t),$$

$$\frac{d}{dt}\,\bar{I}(t)+(\delta+k+\gamma)\,\bar{I}(t) = k\bar{L}.$$

Solving for the IF:

$$\text{IF} = e^{\int(\delta+k+\gamma)\,dt} = e^{(\delta+k+\gamma)t}.$$

Multiply both sides by the IF

$$e^{(\delta+k+\gamma)t}\frac{d}{dt}\,\bar{I}(t)+e^{(\delta+k+\gamma)t}\,(\delta+k+\gamma)\,\bar{I}(t) = k\bar{L}e^{(\delta+k+\gamma)t}.$$

Recalling the product rule $uv'+vu' = \dfrac{d}{dt}(uv)$

$$\frac{d}{dt}\left(e^{(\delta+k+\gamma)t}\bar{I}(t)\right) = k\bar{L}e^{(\delta+k+\gamma)t}.$$

Integrating both sides

$$\int\frac{d}{dt}\big(e^{(\delta+k+\gamma)t}\bar{I}(t)\big) = k\bar{L}\int e^{(\delta+k+\gamma)t}\,dt,$$

$$e^{(\delta+k+\gamma)t}\bar{I}(t) = \frac{k\bar{L}}{(\delta+k+\gamma)}e^{(\delta+k+\gamma)t}+c.$$

Divide both sides by the IF

$$\bar{I}(t) = \frac{k\bar{L}}{(\delta+k+\gamma)}+ce^{-(\delta+k+\gamma)t}.$$

Taking the limit as $t\to\infty$

$$\lim_{t\to\infty}\bar{I}(t) = \frac{k\bar{L}}{\delta+k+\gamma}. \tag{3.17}$$

Consider the fourth equation

$$\frac{d}{dt}\,\bar{R}(t) = \gamma\,\bar{I} + \bar{S}z - (\psi+\delta)\,\bar{R}(t),$$

$$\frac{d}{dt}\,\bar{R}(t) + (\psi+\delta)\bar{R}(t) = \gamma\,\bar{I} + \bar{S}z.$$

Solve for the IF:

$$\text{IF} = e^{\int(\psi+\delta)\,dt} = e^{(\psi+\delta)t}.$$

Multiply both sides by the IF

$$e^{(\psi+\delta)t}\,\frac{d}{dt}\,\bar{R}(t) + e^{(\psi+\delta)t}(\psi+\delta)\bar{R}(t) = \big(\gamma\,\bar{I} + \bar{S}z\big)e^{(\psi+\delta)t}.$$

Recalling the product rule $uv' + vu' = \dfrac{d}{dt}(uv)$

$$\frac{d}{dt}\big(e^{(\psi+\delta)t}\bar{R}\big) = \big(\gamma\,\bar{I} + \bar{S}z\big)e^{(\psi+\delta)t}.$$

Integrate both sides

$$\int \frac{d}{dt}\big(e^{(\psi+\delta)t}\bar{R}\big) = \big(\gamma\,\bar{I} + \bar{S}z\big)\int e^{(\psi+\delta)t}\,dt,$$

$$e^{(\psi+\delta)t}\bar{R} = \frac{\big(\gamma\,\bar{I} + \bar{S}z\big)}{(\psi+\delta)}e^{(\psi+\delta)t} + c.$$

Divide both sides by IF

$$\bar{R} = \frac{\big(\gamma\,\bar{I} + \bar{S}z\big)}{(\psi+\delta)} + ce^{-(\psi+\delta)t}.$$

Taking limit as $t \to \infty$

$$\lim_{t\to\infty}\bar{R} = \frac{\big(\gamma\,\bar{I} + \bar{S}z\big)}{(\psi+\delta)}. \tag{3.18}$$

Considering the second equation

$$\frac{d}{dt}\,\bar{L}(t) = \mu\,\bar{S}\bar{L}(t) - (\delta+k)\bar{L}(t),$$

$$\frac{d}{dt}\,\bar{L}(t) + (\delta+k-\mu\,\bar{S})\bar{L}(t) = 0.$$

Solve for the IF:

$$\text{IF} = e^{(\delta+k-\mu\,\bar{S})\,dt} = e^{(\delta+k-\mu\,\bar{S})t}.$$

Multiply both sides by IF

$$e^{(\delta+k-\mu\,\bar{S})t}\,\frac{d}{dt}\,\bar{L}(t) + e^{(\delta+k-\mu\,\bar{S})t}\big(\delta+k-\mu\,\bar{S}\big)\bar{L}(t) = 0.$$

Recalling product rule

$$\frac{d}{dt}\big(e^{(\delta+k-\mu\,\bar{S})t}\,\bar{L}(t)\big) = 0.$$

Integrating both sides, we obtain

$$\int \frac{d}{dt}\left(e^{(\delta+k-\mu\,\bar{S})t}\bar{L}(t)\right) = 0,$$

$$e^{(\delta+k-\mu\,\bar{S})t}\bar{L}(t) = 0.$$

Dividing both sides by IF, we finally get

$$L(t) = 0. \tag{3.19}$$

From the equations (3.16) – (3.19), $\bar{E}$ is globally asymptotically stable.

4. Conclusion and recommendation. ***4.1. Conclusion.*** This study has developed a deterministic model and utilized the model to subjectively study the transmission flow of tuberculosis contamination. Epidemiological investigations have been performed on the infection model. The study of this model has shown that tuberculosis infection can be stable for both the disease-free and endemic equilibria.

4.2. Recommendation. Based on the findings of the qualitative analysis of the mathematical model of tuberculosis transmission and control, the following recommendations can be made:

(1) *Strengthen early case detection and treatment:* given the significant impact of early case detection and effective treatment on reducing TB transmission, it is recommended to strengthen diagnostic capabilities and ensure timely access to quality TB diagnostic tools and treatment services. This includes expanding laboratory facilities, training healthcare workers on accurate diagnosis, and ensuring availability of appropriate treatment regimens.

(2) *Enhance vaccination programs:* the findings suggest that vaccination programs can contribute to reducing the overall burden of TB. It is recommended to prioritize the implementation of TB vaccination programs, such as Bacille Calmette – Gurin vaccination, particularly in high-burden areas or among vulnerable populations. Ensuring widespread coverage and regular vaccination updates can help mitigate the spread of TB.

(3) *Focus on high-risk populations and settings:* the qualitative analysis identified certain populations and settings that are at a higher risk of TB transmission. It is recommended to prioritize interventions and control measures in these high-risk populations, such as overcrowded communities, healthcare facilities, prisons, and vulnerable populations (e.g., immunocompromised individuals, people living with HIV/AIDS). Targeted interventions can help reduce transmission and prevent outbreaks.

(4) *Continuous monitoring and surveillance:* to assess the effectiveness of control strategies and adapt interventions accordingly, it is recommended to establish robust monitoring and surveillance systems for TB. This includes regular collection and analysis of TB-related data, monitoring of treatment outcomes and assessing the impact of control measures. Such monitoring and surveillance systems enable evidence-based decision-making and early detection of emerging challenges.

(5) *Further research:* the qualitative analysis of the mathematical model provides a foundation for further research. It is recommended to continue refining and validating the model using empirical data, expanding its scope to incorporate additional factors, and exploring the impact of

specific interventions in different contexts. Further research can deepen our understanding of TB dynamics, identify gaps in knowledge, and inform the development of more comprehensive and accurate models.

By implementing these recommendations, policy makers and public health officials can enhance TB control strategies, reduce transmission rates, improve treatment outcomes, and ultimately contribute to the global efforts to eliminate tuberculosis as a public health threat.

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References

1. A. O. Olayinka, O. Anthonia, K. Yetunde, *Prevalence of diabetes mellitus in persons with tuberculosis in a tertiary health centre in Lagos, Nigeria*, Indian J. Endocrinol. Metab., **17(3)**, Article 486-9 (2013).
2. L. N. Nkamba, T. T. Manga, F. Agouanet, M. L. Mann Manyombe, *Mathematical model to assess vaccination and effective contact rate impact in the spread of tuberculosis*, J. Biol. Dyn., **13(1)**, 26 – 42 (2019).
3. U. Aduh et al., *Addressing gaps in adolescent tuberculosis programming and policy in Nigeria from a public health perspective*, Int. J. Adolescent Med. Health, **33(3)**, 41 – 51 (2021).
4. E. N. James, G. D. Miteu, F. N. Ajuk, *A critical review of the response of Nigeria to tuberculosis as an infectious disease and sickle cell as a long-term disease condition*, World J. Biol. Pharmacy Health Sci., **11(2)**, 001 – 008 (2022).
5. M. Rifat et al., *Development of multidrug resistant tuberculosis in Bangladesh: a case-control study on risk factors*, PLOS One, **9(8)**, 105 – 214 (2014).
6. O. Oladimeji et al., *Gender and drug-resistant tuberculosis in Nigeria*, Tropical Med. Infec. Dis., **8(2)**, № 104 (2023).
7. D. P. Gao, N. J. Huang, *Optimal control analysis of a tuberculosis model*, Appl. Math. Model., **58**, 47 – 64 (2018).
8. J. Liu, T. Zhang, *Global stability for a tuberculosis model*, Math. Comput. Model., **54**, Issues 1, 2, 836 – 845 (2011); https://doi.org/10.1016/j.mcm.2011.03.033.
9. K. O. Ugwu, M. C. Agbo, I. M. Ezeonu, *Prevalence of tuberculosis, drug-resistant tuberculosis and HIV/TB co-infection in Enugu, Nigeria*, Afr. J. Infec. Dis., **15(2)**, 24 – 30 (2021).
10. J. R. Andrews et al., *Risk of progression to active tuberculosis following reinfection with mycobacterium tuberculosis*, Clinical Infec. Dis., **54(6)**, 784 – 791 (2012).
11. M. Belay, G. Bjune, G. Ameni, F. Abebe, *Diagnostic and treatment delay among tuberculosis patients in a far region, Ethiopia: a cross-sectional study*, BMC Public Health, **12(1)**, 1 – 8 (2012).
12. H. A. A. Al-Darraji, F. L. Altice, A. Kamarulzaman, *Undiagnosed pulmonary tuberculosis among prisoners in Malaysia: an overlooked risk for tuberculosis in the community*, Tropical Med., Int. Health, **21(8)**, 1049 – 1058 (2016).
13. A. Asefa, W. Teshome, *Total delay in treatment among smear positive pulmonary tuberculosis patients in five primary health centers, Southern Ethiopia: a cross sectional study*, PLOS One, **9(7)**, 102 – 884 (2014).
14. J. C. Blackwood, L. M. Childs, *An introduction to compartmental modeling for the budding infectious disease modeler*, Let. Biomath., **5(1)**, 195 – 221 (2018); https://doi.org/10.30707/LiB5.1Blackwood.
15. A. Egonmwan, D. Okuonghae, *Analysis of a mathematical model for tuberculosis with diagnosis*, J. Appl. Math. Comput., **59**, 129 – 162 (2019).
16. D. Okuonghae, S. Omosigho, *Analysis of a mathematical model for tuberculosis: what could be done to increase case detection*, J. Theor. Biol., **269(1)**, 31 – 45 (2011).

17. D. Okuonghae, *Analysis of a stochastic mathematical model for tuberculosis with case detection*, Int. J. Dyn. Control, **10(3)**, 734 – 747 (2022).

18. J. A. Akingbade, B. S. Ogundare, *Boundedness and stability properties of solutions of mathematical model of measles*, Tamkang J. Math., **52(1)**, 91 – 112 (2021); DOI: 10.5556/j.tkjm.52.2021.3367.

19. K. C. Nwogu, *Determinants of tuberculosis and diabetes mellitus co-morbidity among newly diagnosed tuberculosis patients attending NTBLCP clinics in Abia State*, Public. Health (2017).

20. A. U. Kalu, I. Agwu, A. Agbanyim, *Mathematical analysis of the endemic equilibrium of the transmission dynamics of tuberculosis*, Int. J. Sci. Technology Res., **2(12)**, 263 – 269 (2013).

21. E. X. DeJesus, C. Kaufman, *Routh – Hurwitz criterion in the examination of eigenvalues of a system of nonlinear ordinary differential equations*, Phys. Rev. A, **35(12)**, 52 – 88 (1987).

22. F. A. Ogbo et al., *Tuberculosis disease burden and attributable risk factors in Nigeria*, Tropical Med. Health, **46(1)**, 1 – 11 (2018).

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