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DAMPING OF THE SWIRLING WAVE MODE

ЗАТУХАННЯ РЕЖИМУ КРУГОВИХ ХВИЛЬ

Anton Miliaiev, Alexander Timokha

Institute of Mathematics, National Academy of Sciences of Ukraine
3 Tereshchenkivska St., Kyiv, 01024, Ukraine
e-mail: amilyaev@gmail.com
atimokha@gmail.com, corresponding author

We generalize recent results of the authors [A. Miliaev and A. Timokha, *Viscous damping of steady-state resonant sloshing in a clean rectangular tank*, J. Fluid Mech., **965**, № R1, 1 – 11 (2023); DOI: doi.org/10.1017/jfm.2023.372] on machine learning of multidimensional modal systems derived to describe nonlinear sloshing in rigid tanks exposed to resonant excitations. The multimodal systems neglect viscous damping but the learning procedure pursues incorporating a priori unknown damping terms into them. The main focus is on steady-state swirling wave regime in an upright circular tank which moves longitudinally by a harmonic law with the excitation frequency close to the lowest sloshing eigenfrequency. The nonlinear Narimanov – Moiseev-type modal system is employed with a set of phenomenological linear and nonlinear damping quantities whose coefficients are a subject of the learning procedure. The procedure requires semi-analytical periodic (steady-state) solutions of the Narimanov – Moiseev-type system which are constructed in the present paper. Based on these solutions and the loss (cost) function expressing the “distance” between measured and theoretical phase-lags for the swirling wave regime, the constructed learning procedure is tested with experimental data by A. Royon-Lebeaud, E. J. Hopfinger, and A. Cartellier [*Liquid sloshing and wave breaking in circular and square-base cylindrical containers*, J. Fluid Mech., **577**, 467 – 494 (2007); DOI: doi.org/10.1017/S0022112007004764].

Узагальнено останні результати авторів [A. Miliaev, A. Timokha, *Viscous damping of steady-state resonant sloshing in a clean rectangular tank*, J. Fluid Mech., **965**, № R1, 1 – 11 (2023); DOI: doi.org/10.1017/jfm.2023.372] щодо машинного навчання багатовимірних модальних систем, отриманих для опису нелінійного хлюпання рідини в жорстких резервуарах, підданих резонансним збудженням. Мультимодальні системи нехтують в’язким демпфуванням, але навчання передбачає включення в них апріорі невідомих членів, які відповідають такому демпфуванню. Основну увагу приділілено режиму кругової усталеної хвилі у вертикальному круговому резервуарі, який рухається поздовжньо за гармонійним законом із частотою збудження, близькою до найнижчої власної частоти хлюпання. Використано нелінійну модальну систему типу Наріманова – Моісеєва з набором феноменологічних лінійних і нелінійних демпфуючих членів, визначення коефіцієнтів яких є предметом процедури навчання. Процедура потребує напіваналітичних періодичних (стаціонарних) розв’язків системи типу Наріманова – Моісеєва, які побудовано в цій статті. На основі цих розв’язків і функції втрат (витрат), яка виражає “відстань” між виміряними та теоретичними фазовими зсувами для режиму кругової хвилі, розроблену процедуру навчання протестовано за допомогою експериментальних даних A. Royon-Lebeaud, E. J. Hopfinger, A. Cartellier [*Liquid sloshing and wave breaking in circular and square-base cylindrical containers*, J. Fluid Mech., **577**, 467 – 494 (2007); DOI: doi.org/10.1017/S0022112007004764].

1. Introduction. If a clean rigid tank is resonantly excited with the forcing frequency close to the lowest natural sloshing frequency, the free-surface nonlinearity causes a kinetic energy flow from the primary excited to higher natural sloshing modes [1, 2]. Along with viscous damping, the nonlinearity plays significant role to prevent infinite resonant wave response. That is why, semi-analytical nonlinear modal theories whose derivations adopt inviscid-fluid-flow hydrodynamic model [1] provide satisfactory agreement with the measured steady-state wave-amplitudes. However, these theories are inapplicable to description of the phase-lag response (between excitation and periodic wave), which becomes theoretically piecewise function of the forcing frequency in the inviscid approximation but any non-zero damping in the hydrodynamic system makes the dependence continuous. Using the measured phase-lags looks therefore the best way to restore appropriate damping terms in the modal theories as it was done by the machine learning procedure in [3] for a rigid rectangular tank. In the present paper, the Narimanov – Moiseev-type nonlinear modal system [4] and measurements [5] are adopted to generalise the latter results for resonant sloshing in an upright circular cylindrical tank.

2. Statement of the problem. We consider an upright circular base rigid tank of the radius R_0 performing small-amplitude (relative to the radius) oscillatory periodic translatory motions in the horizontal plane that are governed by the generalised time-periodic coordinate $\eta_1(t)$. The excitation period is close to the higher natural sloshing period so that the contained liquid is in resonance, that is, in particular, that the surface-wave magnitude is much larger that the excitation amplitude.

The free surface $\Sigma(t)$ admits the single-valued representation with respect to the vertical coordinate z and liquid flows inside the liquid domain $Q(t)$ are described by the nondimensional velocity potential Φ. The two unknowns $z = \zeta(r, \theta, t)$ and $\Phi(r, \theta, z)$ are defined in the cylindrical coordinate system (r, θ, z) rigidly fixed with the tank so that $z = 0$ coincides with the unperturbed (static) free surface Σ_0 and Oz is parallel (counter-directed) to the gravity acceleration $\boldsymbol{g}$.

The so-called nonlinear multimodal analysis of the resonant liquid sloshing is normally performed in the nondimensional statement suggesting the characteristic dimension R_0 (the free-surface radius) and time $T = 2\pi/\sigma$, where σ is the excitation frequency. When following [4, 6, 7], the multimodal method adopts the free-surface representation by the natural sloshing modes [1, 8, 9]:

$$z = \zeta(r, \theta, t) = \sum_{M,i}^{I_\theta, I_r} \mathcal{R}_{Mi}(r)\cos(M\theta)\, p_{Mi}(t) + \sum_{m,i}^{I_\theta, I_r} \mathcal{R}_{mi}(r)\sin(m\theta)\, r_{mi}(t), \quad I_\theta, I_r \to \infty, \quad (1)$$

where $p_{Mi}(t)$ and $r_{Mi}(t)$ play the role of the generalised hydrodynamic coordinates and the radial component of the natural free-surface modes,

$$\mathcal{R}_{Mi}(r) = \alpha_{Mi} J_M(k_{Mi} r) \quad (2)$$

($J_M(\cdot)$ is the Bessel function of the first kind), depends on the radial wave numbers k_{Mi} coming from equations $J'_M(k_{Mi}) = 0$ (these express no flows through the wetted tank walls) and the normalising multipliers α_{Mi} computed from orthogonality condition,

$$\int_0^1 r\, \mathcal{R}_{Mi}(r)\, \mathcal{R}_{Mj}(r)\, \mathrm{d}r = \delta_{ij}, \quad i, j = 1, \ldots \quad (3)$$

(δ_{ij} is the Kronecker delta), which deduces [6, 7]

$$\alpha_{Mi}^2 = \frac{2k_{Mi}^2}{J_M^2(k_{Mi})\big(k_{Mi}^2 - M^2\big)}. \tag{4}$$

The corresponding dimensional natural sloshing frequencies take the form [1, 7]

$$\sigma_{Mi} = \sqrt{\frac{\kappa_{Mi}\, g}{R_0}}, \quad \kappa_{Mi} = k_{Mi}\tanh(k_{Mi}h), \quad M = 0, \ldots, \quad i = 1, \ldots, \tag{5}$$

where g is the gravity acceleration and h is the R_0-scaled mean liquid depth. The nondimensional natural sloshing frequencies are as follows:

$$\bar{\sigma}_{Mi} = \frac{\sigma_{Mi}}{\sigma}. \tag{6}$$

The generalised hydrodynamic coordinates $p_{Mi}(t)$ and $r_{Mi}(t)$ can be found out after substituting (1) into either the corresponding free-surface boundary problem or its equivalent variational formulation (see, surveys on how it can be done in [1, 6]). These analytical approaches lead to multidimensional systems of ordinary differential equations with respect to the generalised hydrodynamic coordinates.

Combining the Miles – Lukovsky variational and Narimanov – Moiseev asymptotic approaches, the works [4, 7] derived a rather compact weakly nonlinear system of ordinary differential equations. This Narimanov – Moiseev-type modal system is applicable when the excitation frequency σ is close to the lowest natural sloshing frequency σ_{11}, i.e., $\bar{\sigma}_{11} \to 1,$ the nondimensional liquid depth $1.02 \lesssim h$ (this provides no secondary resonances [1, 4, 10] in the hydrodynamic system) and the nondimensional (R_0-scaled) excitation amplitude is small

$$\eta_1 = O(\epsilon) \ll 1. \tag{7}$$

Within the framework of the Narimanov – Moiseev-type multimodal modelling, the nondimensional generalised hydrodynamic coordinates satisfy the asymptotic relations

$$\begin{gathered} p_{11} \sim r_{11} = O\big(\epsilon^{1/3}\big), \quad p_{0i} \sim p_{2i} \sim r_{2i} = O\big(\epsilon^{2/3}\big), \\ p_{3i} \sim r_{3i} \sim p_{1\,i+1} \sim r_{1\,i+1} = O(\epsilon), \quad i \geq 1, \end{gathered} \tag{8}$$

but the modal system neglects the $o(\epsilon)$ terms and takes the following form [4, 7]:

$$\begin{aligned} &\ddot{p}_{11} + \boxed{2\xi_{11}\bar{\sigma}_{11}\dot{p}_{11} + \mathcal{Q}_{11}(\dot{p}_{11})} + \bar{\sigma}_{11}^2 p_{11} + d_1 p_{11}\big(\ddot{p}_{11}p_{11} + \ddot{r}_{11}r_{11} + \dot{p}_{11}^2 + \dot{r}_{11}^2\big) \\ &\quad + d_2[r_{11}(\ddot{p}_{11}r_{11} - \ddot{r}_{11}p_{11}) + 2\dot{r}_{11}(\dot{p}_{11}r_{11} - \dot{r}_{11}p_{11})] \\ &\quad + \sum_{j=1}^{I_r}\Big[d_3^{(j)}(\ddot{p}_{11}p_{2j} + \ddot{r}_{11}r_{2j} + \dot{p}_{11}\dot{p}_{2j} + \dot{r}_{11}\dot{r}_{2j}) \\ &\quad + d_4^{(j)}(\ddot{p}_{2j}p_{11} + \ddot{r}_{2j}r_{11}) + d_5^{(j)}(\ddot{p}_{11}p_{0j} + \dot{p}_{11}\dot{p}_{0j}) + d_6^{(j)}\ddot{p}_{0j}p_{11}\Big] = -\ddot{\eta}_1\kappa_{11}P_1, \end{aligned} \tag{9a}$$

$$\ddot{r}_{11} + \boxed{2\xi_{11}\bar{\sigma}_{11}\dot{r}_{11} + \mathcal{Q}_{11}(\dot{r}_{11})} + \bar{\sigma}_{11}^2 r_{11} + d_1 r_{11}\big(\ddot{p}_{11}p_{11} + \ddot{r}_{11}r_{11} + \dot{p}_{11}^2 + \dot{r}_{11}^2\big)$$
$$+ d_2\big[p_{11}(\ddot{r}_{11}p_{11} - \ddot{p}_{11}r_{11}) + 2\dot{p}_{11}(\dot{r}_{11}p_{11} - \dot{p}_{11}r_{11})\big]$$
$$+ \sum_{j=1}^{I_r}\Big[d_3^{(j)}(\ddot{p}_{11}r_{2j} - \ddot{r}_{11}p_{2j} + \dot{p}_{11}\dot{r}_{2j} - \dot{p}_{2j}\dot{r}_{11})$$
$$+ d_4^{(j)}(\ddot{r}_{2j}p_{11} - \ddot{p}_{2j}r_{11}) + d_5^{(j)}(\ddot{r}_{11}p_{0j} + \dot{r}_{11}\dot{p}_{0j}) + d_6^{(j)}\ddot{p}_{0j}r_{11}\Big] = 0, \qquad (9b)$$

$$\ddot{p}_{2k} + \boxed{2\xi_{2k}\bar{\sigma}_{2k}\dot{p}_{2k}} + \bar{\sigma}_{2k}^2 p_{2k} + d_{7,k}\big(\dot{p}_{11}^2 - \dot{r}_{11}^2\big) + d_{9,k}(\ddot{p}_{11}p_{11} - \ddot{r}_{11}r_{11}) = 0, \qquad (10a)$$

$$\ddot{r}_{2k} + \boxed{2\xi_{2k}\bar{\sigma}_{2k}\dot{r}_{2k}} + \bar{\sigma}_{2k}^2 r_{2k} + 2d_{7,k}\dot{p}_{11}\dot{r}_{11} + d_{9,k}(\ddot{p}_{11}r_{11} + \ddot{r}_{11}p_{11}) = 0, \qquad (10b)$$

$$\ddot{p}_{0k} + \boxed{2\xi_{0k}\bar{\sigma}_{0k}\dot{p}_{0k}} + \bar{\sigma}_{0k}^2 p_{0k} + d_{8,k}\big(\dot{p}_{11}^2 + \dot{r}_{11}^2\big) + d_{10,k}(\ddot{p}_{11}p_{11} + \ddot{r}_{11}r_{11}) = 0, \qquad (10c)$$

$$\ddot{p}_{3k} + \boxed{2\xi_{3k}\bar{\sigma}_{3k}\dot{p}_{3k}} + \bar{\sigma}_{3k}^2 p_{3k} + d_{11,k}\big[\ddot{p}_{11}\big(p_{11}^2 - r_{11}^2\big) - 2p_{11}r_{11}\ddot{r}_{11}\big]$$
$$+ d_{12,k}\big[p_{11}\big(\dot{p}_{11}^2 - \dot{r}_{11}^2\big) - 2r_{11}\dot{p}_{11}\dot{r}_{11}\big] + \sum_{j=1}^{I_r}\Big[d_{13,k}^{(j)}(\ddot{p}_{11}p_{2j} - \ddot{r}_{11}r_{2j})$$
$$+ d_{14,k}^{(j)}(\ddot{p}_{2j}p_{11} - \ddot{r}_{2j}r_{11}) + d_{15,k}^{(j)}(\dot{p}_{2j}\dot{p}_{11} - \dot{r}_{2j}\dot{r}_{11})\Big] = 0, \qquad (11a)$$

$$\ddot{r}_{3k} + \boxed{2\xi_{3k}\bar{\sigma}_{3k}\dot{r}_{3k}} + \bar{\sigma}_{3k}^2 r_{3k} + d_{11,k}\big[\ddot{r}_{11}\big(p_{11}^2 - r_{11}^2\big) + 2p_{11}r_{11}\ddot{p}_{11}\big]$$
$$+ d_{12,k}\big[r_{11}\big(\dot{p}_{11}^2 - \dot{r}_{11}^2\big) + 2p_{11}\dot{p}_{11}\dot{r}_{11}\big] + \sum_{j=1}^{I_r}\Big[d_{13,k}^{(j)}(\ddot{p}_{11}r_{2j} + \ddot{r}_{11}p_{2j})$$
$$+ d_{14,k}^{(j)}(\ddot{p}_{2j}r_{11} + \ddot{r}_{2j}p_{11}) + d_{15,k}^{(j)}(\dot{p}_{2j}\dot{r}_{11} + \dot{r}_{2j}\dot{p}_{11})\Big] = 0, \quad k = 1, \ldots, I_r, \qquad (11b)$$

$$\ddot{p}_{1n} + \boxed{2\xi_{1n}\bar{\sigma}_{1n}\dot{p}_{1n}} + \bar{\sigma}_{1n}^2 p_{1n} + d_{16,n}\big(\ddot{p}_{11}p_{11}^2 + r_{11}p_{11}\ddot{r}_{11}\big) + d_{17,n}\big(\ddot{p}_{11}r_{11}^2 - r_{11}p_{11}\ddot{r}_{11}\big)$$
$$+ d_{18,n}p_{11}\big(\dot{p}_{11}^2 + \dot{r}_{11}^2\big) + d_{19,n}\big(r_{11}\dot{p}_{11}\dot{r}_{11} - p_{11}\dot{r}_{11}^2\big)$$
$$+ \sum_{j=1}^{I_r}\Big[d_{20,n}^{(j)}(\ddot{p}_{11}p_{2j} + \ddot{r}_{11}r_{2j}) + d_{21,n}^{(j)}(p_{11}\ddot{p}_{2j} + r_{11}\ddot{r}_{2j}) + d_{22,n}^{(j)}(\dot{p}_{11}\dot{p}_{2j} + \dot{r}_{11}\dot{r}_{2j})$$
$$+ d_{23,n}^{(j)}\ddot{p}_{11}p_{0j} + d_{24,n}^{(j)}p_{11}\ddot{p}_{0j} + d_{25,n}^{(j)}\dot{p}_{11}\dot{p}_{0j}\Big] = -\ddot{\eta}_1\kappa_{1n}P_n, \qquad (12a)$$

$$\ddot{r}_{1n} + \boxed{2\xi_{1n}\bar{\sigma}_{1n}\dot{r}_{1n}} + \bar{\sigma}_{1n}^2 r_{1n} + d_{16,n}\big(\ddot{r}_{11}r_{11}^2 + r_{11}p_{11}\ddot{p}_{11}\big) + d_{17,n}\big(\ddot{r}_{11}p_{11}^2 - r_{11}p_{11}\ddot{p}_{11}\big)$$
$$+ d_{18,n}r_{11}\big(\dot{p}_{11}^2 + \dot{r}_{11}^2\big) + d_{19,n}\big(p_{11}\dot{p}_{11}\dot{r}_{11} - r_{11}\dot{p}_{11}^2\big)$$
$$+ \sum_{j=1}^{I_r}\Big[d_{20,n}^{(j)}(\ddot{p}_{11}r_{2j} - \ddot{r}_{11}p_{2j}) + d_{21,n}^{(j)}(p_{11}\ddot{r}_{2j} - r_{11}\ddot{p}_{2j}) + d_{22,n}^{(j)}(\dot{p}_{11}\dot{r}_{2j} - \dot{r}_{11}\dot{p}_{2j})$$

$$+\, d^{(j)}_{23,n}\ddot{r}_{11}p_{0j} + d^{(j)}_{24,n}r_{11}\ddot{p}_{0j} + d^{(j)}_{25,n}\dot{r}_{11}\dot{p}_{0j}\Big] = 0, \quad n = 2,\ldots,I_r, \tag{12b}$$

where

$$P_j = \frac{1}{k_{1j}}\sqrt{\frac{2}{k_{1j}^2 - 1}} \tag{13}$$

and the nondimensional hydrodynamic coefficients at the nonlinear quantities are the given functions of the nondimensional liquid depth h which are tabled in [7].

Following [3, 7], the nonlinear Narimanov – Moiseev-type equations (9) – (12) are equipped with extra framed terms which are not derivable from the original mathematical statement but are needed to account for viscous damping in the hydrodynamical system. The terms include linear components with the damping coefficients $2\xi_{Mi}\bar{\sigma}_{Mi}$ ($0 < \xi_{Mi} \ll 1$ are the viscous damping rates) which can physically be related to logarithmic decrements of the natural sloshing modes when the free-surface nonlinearity does not matter. In addition, the nonlinear damping quantity $\mathcal{Q}_{11}$ is introduced in (9) governing the lowest-order (dominant) generalised hydrodynamic coordinates $p_{11}(t)$ and $r_{11}(t)$.

Experimental estimates of the damping rates ξ_{Mi} might be done by measuring the logarithmic decrements [1, 11, 12] of the corresponding natural sloshing modes (standing waves). The procedure of restoring ξ_{Mi} from these measurements could mathematically be interpreted as a [machine] learning of the linearized modal equations (9) – (12). Theoretical predictions of ξ_{Mi} are also derivable by solving a spectral boundary value problem on linear sloshing of a viscous liquid [13, 14]. Alternatively, the theoretical damping rates ξ_{Mi} can be estimated by employing asymptotic formulas by Keulegan [7, 11, 12, 15] that assumes a laminar viscous flow near the wetted tank surface.

Specifically, all the aforementioned approaches to estimating the damping rates ξ_{Mi} are invalid for nonlinear resonant sloshing when total energy loss is seriously affected by energy flow from lower to higher natural sloshing modes [2]. This means that ξ_{Mi} in (9) – (12) are poorly predicted by existing theories [7, 11 – 13, 15]. The latter is also true for experiments when the free-surface nonlinearity is not negligible.

The nonlinear Narimanov – Moiseev-type modal system (10) – (12) is of asymptotic nature. It neglects the $o(\epsilon)$-order quantities including whose associated with viscous damping. Furthermore, one should recall that the damping rates ξ_{Mi} must be small values. It is because their values should be proportional to the viscous boundary layer thickness at the liquid boundary but the thickness must be small if an inviscid-liquid hydrodynamic model is adopted.

In [7], the damping rates ξ_{Mi} are assumed being of the $O(\epsilon^{2/3})$-order that made it possible to completely neglect viscous damping in (10) – (12) and require $\mathcal{Q}_{11} \equiv 0$. Under this asymptotic assumption, the Narimanov – Moiseev-type modal theory completely fails to fit the measured data in [5] on the phase lag for the steady-state swirling wave mode. On the other hand, the authors [3] showed that assuming the $O(\epsilon^{1/3})$-order damping rates in the Narimanov – Moiseev-type modal equations for sloshing in a rectangular tank makes it possible to fit analogous measurements in [16]. Let us further follow [3] and, therefore, postulate $\xi_{Mi} = O(\epsilon^{1/3})$ and

$$\mathcal{Q}_{11}(f) = \xi_1\,\frac{3\pi}{4}\,f|f| + o(f^2), \quad \xi_1 = O(\epsilon^{1/3}). \tag{14}$$

3. Steady-state waves. The nonlinear modal equations (9) – (12) were derived assuming resonant excitation of the lowest natural sloshing mode. Consider the harmonic resonant longitudinal forcing of a small amplitude, i.e.,

$$\eta_1(t) = \eta_{1a} \cos t, \quad \eta_{1a} = O(\epsilon) \ll 1, \tag{15}$$

when the excitation (equal to the characteristic) frequency σ is close to the first natural sloshing frequency σ_{11} as well as the Moiseev asymptotic detuning is satisfied,

$$\Lambda = \bar{\sigma}_{11}^2 - 1 = \frac{\sigma_{11}^2}{\sigma^2} - 1 = O\big(\epsilon^{2/3}\big). \tag{16}$$

Even though the Narimanov – Moiseev-type equations (9) – (12) are equipped with extra damping terms, its asymptotic periodic solution can be constructed by combining asymptotic and Fourier harmonic analysis from [4, 7]. The starting point consists of considering the first (lowest) order asymptotic approximation which is associated with amplitudes at the lowest Fourier harmonics in the dominant generalised hydrodynamic coordinates

$$p_{11}(t) = a \cos t + \bar{a} \sin t + O(\epsilon), \qquad r_{11}(t) = \bar{b} \cos t + b \sin t + O(\epsilon), \tag{17}$$

where the nondimensional amplitude parameters $a,\ \bar{a},\ \bar{b}$, and $b = O\big(\epsilon^{1/3}\big)$. Why these four lowest order amplitudes are of the lowest asymptotic order $O\big(\epsilon^{1/3}\big)$ in the Narimanov – Moiseev-type approximation is explained in [4, 7].

Substitution of (17) into (10) derives

$$p_{0k}(t) = s_{0k}\big(a^2 + \bar{a}^2 + b^2 + \bar{b}^2\big) + s_{1k}\big[\big(a^2 - \bar{a}^2 - b^2 + \bar{b}^2\big) \cos 2t + 2\big(a\bar{a} + b\bar{b}\big) \sin 2t\big]$$
$$+ s_{2k}\big[-2\big(a\bar{a} + b\bar{b}\big) \cos 2t + \big(a^2 - \bar{a}^2 - b^2 + \bar{b}^2\big) \sin 2t\big] + o(\epsilon), \tag{18a}$$

$$p_{2k}(t) = c_{0k}\big(a^2 + \bar{a}^2 - b^2 - \bar{b}^2\big) + c_{1k}\big[\big(a^2 - \bar{a}^2 + b^2 - \bar{b}^2\big) \cos 2t + 2\big(a\bar{a} - b\bar{b}\big) \sin 2t\big]$$
$$+ c_{2k}\big[-2\big(a\bar{a} - b\bar{b}\big) \cos 2t + \big(a^2 - \bar{a}^2 + b^2 - \bar{b}^2\big) \sin 2t\big] + o(\epsilon), \tag{18b}$$

$$r_{2k}(t) = 2c_{0k}\big(a\bar{b} + b\bar{a}\big) + 2c_{1k}\big[\big(a\bar{b} - b\bar{a}\big) \cos 2t + \big(ab + \bar{a}\bar{b}\big) \sin 2t\big]$$
$$+ 2c_{2k}\big[-\big(ab + \bar{a}\bar{b}\big) \cos 2t + \big(a\bar{b} - b\bar{a}\big) \sin 2t\big] + o(\epsilon), \tag{18c}$$

where

$$s_{0k} = \frac{d_{10,k} - d_{8,k}}{2\bar{\sigma}_{0k}^2}, \qquad s_{1k} = \frac{d_{10,k} + d_{8,k}}{2\Delta_{0k}}\big(\bar{\sigma}_{0k}^2 - 4\big), \qquad s_{2k} = 2\,\frac{d_{10,k} + d_{8,k}}{\Delta_{0k}}\,\xi_{0,k}\bar{\sigma}_{0k},$$
$$c_{0k} = \frac{d_{9,k} - d_{7,k}}{2\bar{\sigma}_{2k}^2}, \qquad c_{1k} = \frac{d_{9,k} + d_{7,k}}{2\Delta_{2k}}\big(\bar{\sigma}_{2k}^2 - 4\big), \qquad c_{2k} = 2\,\frac{d_{9,k} + d_{7,k}}{\Delta_{2k}}\,\xi_{2,k}\bar{\sigma}_{2k} \tag{19}$$

and

$$\Delta_{0k} = \big(\bar{\sigma}_{0k}^2 - 4\big)^2 + 16\xi_{0k}^2\bar{\sigma}_{0k}^2, \qquad \Delta_{2k} = \big(\bar{\sigma}_{2k}^2 - 4\big)^2 + 16\xi_{2k}^2\bar{\sigma}_{2k}^2. \tag{20}$$

Inserting (17) and (18) into (9) and collecting the terms in front of the first Fourier harmonics, $\cos t$ and $\sin t$, deduce the necessary solvability (secular) conditions

$$a\big[\Lambda + m_1\big(a^2+\bar a^2+\bar b^2\big) + m_3 b^2 - 2(m_7-m_6)b\bar b\big] + \bar a\Big[(m_1-m_3)\bar b\, b + (\xi_0+\xi_1 A) - (m_6+m_7)\big(a^2+\bar a^2+b^2\big) - (3m_6-m_7)\bar b^2\Big] = \epsilon_x = \kappa_{11}P_1\eta_{1a}, \tag{21a}$$

$$\bar a\big[\Lambda + m_1\big(a^2+\bar a^2+b^2\big) + m_3\bar b^2 + 2(m_7-m_6)b\bar b\big] + a\big[(m_1-m_3)\bar b b - (\xi_0+\xi_1 A) + (m_6+m_7)\big(a^2+\bar a^2+\bar b^2\big) + (3m_6-m_7)b^2\big] = 0, \tag{21b}$$

$$b\big[\Lambda + m_1\big(b^2+\bar b^2+\bar a^2\big) + m_3 a^2 + 2(m_7-m_6)a\bar a\big] + \bar b\big[(m_1-m_3)\bar a a - (\xi_0+\xi_1 B) + (m_6+m_7)\big(b^2+\bar b^2+a^2\big) + (3m_6-m_7)\bar a^2\big] = 0, \tag{21c}$$

$$\bar b\big[\Lambda + m_1\big(b^2+\bar b^2+a^2\big) + m_3\bar a^2 - 2(m_7-m_6)a\bar a\big] + b\big[(m_1-m_3)\bar a a(\xi_0+\xi_1 B) - (m_6+m_7)\big(b^2+\bar b^2+\bar a^2\big) - (3m_6-m_7)a^2\big] = 0 \tag{21d}$$

with respect to $a,\ \bar a,\ \bar b$ and $b,\ \xi_0 = 2\xi_{11},$ so that the system (21) defines the amplitude parameters as functions the frequency parameter Λ by (16). The coefficients m_i are calculated by the formulas

$$m_1 = -\frac{1}{2}\,d_1 + \sum_{j=1}^{I_r}\left[c_{1j}\left(\frac{1}{2}\,d_3^{(j)} - 2d_4^{(j)}\right) + s_{1j}\left(\frac{1}{2}\,d_5^{(j)} - 2d_6^{(j)}\right) - s_{0j}d_5^{(j)} - c_{0j}d_3^{(j)}\right], \tag{22a}$$

$$m_3 = \frac{1}{2}\,d_1 - 2d_2 + \sum_{j=1}^{I_r}\left[c_{1j}\left(\frac{3}{2}\,d_3^{(j)} - 6d_4^{(j)}\right) + s_{1j}\left(-\frac{1}{2}\,d_5^{(j)} + 2d_6^{(j)}\right) - s_{0j}d_5^{(j)} + c_{0j}d_3^{(j)}\right], \tag{22b}$$

$$m_6 = \sum_{j=1}^{I_r} c_{2j}\left(\frac{1}{2}\,d_3^{(j)} - 2d_4^{(j)}\right), \qquad m_7 = \sum_{j=1}^{I_r} s_{2j}\left(\frac{1}{2}\,d_5^{(j)} - 2d_6^{(j)}\right), \tag{22c}$$

which are formally functions of h and $\bar\sigma_{11} = 1 + O\big(\epsilon^{2/3}\big)$ so that excluding the higher-order terms in m_k implies considering $\bar\sigma_{0k}^2$ and $\bar\sigma_{2k}^2$ in expressions (19) and (20) independent of σ^2 and equal to $\sigma_{0k}^2/\sigma_{11}^2$ and $\sigma_{2k}^2/\sigma_{11}^2$, respectively. Details on why this can be done in the resonant case are reported in [7].

Following [17], we introduce the integral amplitudes

$$A = \sqrt{a^2+\bar a^2} \qquad \text{and} \qquad B = \sqrt{\bar b^2 + b^2} > 0, \tag{23}$$

which link the original amplitude parameters as follows

$$a = A\cos\psi, \qquad \bar a = A\sin\psi, \qquad \bar b = B\cos\varphi, \qquad b = B\sin\varphi, \tag{24}$$

where ψ and φ are the phase lags.

By substituting (24) into expressions $[\bar{a}\,(21a) - a\,(21b)]$, $[\bar{b}\,(21c) - b\,(21d)]$, $[a\,(21a) + \bar{a}\,(21b)]$, and $[b\,(21c) + \bar{b}\,(21d)]$, we arrive at the following alternative secular equations:

$$\begin{cases} \boxed{1}: A\big[\Lambda + m_1 A^2 + (\mathcal{F} + \mathcal{H})B^2\big] = \epsilon_x \cos\psi, \\ \boxed{2}: B\big[\Lambda + m_1 B^2 + (\mathcal{F} - \mathcal{H})A^2\big] = 0, \\ \boxed{3}: A\big[(\mathcal{D} + \mathcal{G})B^2 + (\xi_0 + \xi_1 A) - (m_6 + m_7)A^2\big] = \epsilon_x \sin\psi, \\ \boxed{4}: B\big[(\mathcal{D} - \mathcal{G})A^2 - (\xi_0 + \xi_1 B) + (m_6 + m_7)B^2\big] = 0, \end{cases} \tag{25}$$

where

$$\mathcal{F} = \mathcal{F}_\alpha = m_1 \cos^2\alpha + m_3 \sin^2\alpha, \qquad \mathcal{D} = \mathcal{D}_\alpha = (m_3 - m_1)\sin\alpha\cos\alpha,$$
$$\mathcal{G} = \mathcal{G}_\alpha = m_7 - 3m_6 + 2(m_6 - m_7)\cos^2\alpha, \qquad \mathcal{H} = \mathcal{H}_\alpha = 2(m_6 - m_7)\sin\alpha\cos\alpha, \tag{26}$$

and $\alpha = \varphi - \psi$. The secular systems (21) and (25) are mathematically equivalent. Having known $A,\ B,\ \psi,\ \varphi$ from (25), we can define $a,\ \bar{a},\ b,\ \bar{b}$ and vice versa. There are two physically different solutions which determine planar and swirling steady-state waves.

The *planar* steady-state waves correspond to $B = 0$ and $A > 0$ (phase lag φ is not defined), where A and ψ are analytically represented by

$$A^2\Big[\big(\Lambda + m_1 A^2\big)^2 + \big(\xi_0 + \xi_1 A - (m_6 + m_7)A^2\big)^2\Big] = \epsilon_x^2,$$
$$\Updownarrow$$
$$-1 < \Lambda_{[A]} = \pm\sqrt{\frac{\epsilon_x^2}{A^2} - (\xi_0 + \xi_1 A - (m_6 + m_7)A^2)^2} - m_1 A^2, \tag{27}$$
$$\psi_{[A]} = \operatorname{atan2}\left(\frac{A\big(\xi_0 + \xi_1 A - (m_6 + m_7)A^2\big)}{\epsilon_x}, \frac{A\big(\Lambda_{[A]} + m_1 A^2\big)}{\epsilon_x}\right),$$

which parametrically determine the wave amplitude $\big(\sigma/\sigma_1 = \big(\Lambda_{[A]} + 1\big)^{-1/2}, A, 0\big)$ and phase lag $\big(\sigma/\sigma_1 = (\Lambda_{[A]} + 1)^{-1/2}, \psi_{[A]}, 0\big)$ response curves as functions of $A_{\min} < A < A_{\max}$, where $A_{\max}$ comes from the positiveness of expression under the square root and $A_{\min}$ is associated with the equality $-1 < \Lambda_{[A]}$.

The *swirling* steady-state wave regime is associated with solution of (25) when $B > 0$. To get this solution, the secular system should be rewritten in the form

$$\boxed{4}: A^2_{[B,\alpha]} = \frac{\xi_0 + \xi_1 B - (m_6 + m_7)B^2}{\mathcal{D}_\alpha - \mathcal{G}_\alpha} > 0, \tag{28a}$$

$$\boxed{2}: \Lambda_{[B,\alpha]} = -m_1 B^2 - (\mathcal{F}_\alpha - \mathcal{H}_\alpha)A^2_{[B,\alpha]} > -1, \tag{28b}$$

$$\boxed{1}^2 + \boxed{4}^2: \epsilon_x^2 = A^2_{[B,\alpha]}\bigg(\Big[\Lambda_{[B,\alpha]} + m_1 A^2_{[B,\alpha]} + (\mathcal{F}_\alpha + \mathcal{H}_\alpha)B^2\Big]^2$$
$$+\Big[\big(\xi_0 + \xi_1 A_{[B,\alpha]}\big) - (m_6 + m_7)A^2_{[B,\alpha]} + (\mathcal{D}_\alpha + \mathcal{G}_\alpha)B^2\Big]^2\bigg), \tag{28c}$$

$$\boxed{1},\ \boxed{4}:\ \psi_{[B,\alpha]} = \operatorname{atan}2\left(\frac{A_{[B,\alpha]}\Big(\xi_0 + \xi_1 A_{[B,\alpha]} - (m_6+m_7)A^2_{[B,\alpha]}\Big)}{\epsilon_x},\right.$$

$$\left.\frac{A_{[B,\alpha]}\Big(\Lambda_{[B,\alpha]} + m_1 A^2_{[B,\alpha]}\Big)}{\epsilon_x}\right), \tag{28d}$$

$$\varphi_{[B,\alpha]} = \alpha + \psi_{[B,\alpha]}, \tag{28e}$$

whose consequently usage makes it possible to draw the amplitude/phase-lag response branchings, parametrically, as functions of B.

Getting the branchings implies the following computational procedure:

1°. Consider a trial fixed value $B > 0$.

2°. Because $\mathcal{D}_\alpha,\ \mathcal{G}_\alpha,\ \mathcal{F}_\alpha$, and $\mathcal{H}_\alpha$ are the π-periodic functions of α, one can find intervals $(\alpha_i, \alpha_{i+1}),\ 0 \le \alpha_i < \pi$, where inequality (28a) is satisfied for the chosen B and, therefore, the real $A > 0$ exists.

3°. Substitute (28a) and (28b) into (28c) to get (for the chosen fixed B) an equation with respect to α defined on the above computed intervals $(\alpha_i, \alpha_{i+1}),\ 0 \le \alpha_i < \pi$. Denote these roots as $\alpha^{[k]}_{[B]},\ k = 1, \ldots, N_B$.

4°. Consequently inserting B and $\alpha = \alpha^{[k]}_{[B]},\ k = 1, \ldots, N_B$, into (28a), (28b), and (28e) determines N_B points on the amplitude

$$\left(\sigma/\sigma_1 = \left(\Lambda_{\left[B,\alpha^{[k]}_{[B]}\right]} + 1\right)^{-1/2}, A_{\left[B,\alpha^{[k]}_{[B]}\right]}, B\right), \quad k = 1, \ldots, N_B, \tag{29}$$

and phase-lag

$$\left(\sigma/\sigma_1 = \left(\Lambda_{\left[B,\alpha^{[k]}_{[B]}\right]} + 1\right)^{-1/2}, \psi_{\left[B,\alpha^{[k]}_{[B]}\right]}, \varphi_{\left[B,\alpha^{[k]}_{[B]}\right]}\right), \quad k = 1, \ldots, N_B, \tag{30}$$

response curves.

5°. Varying B between 0 and $B_{\max}$ which obviously exists since the right-hand side of (28c) increases with increasing B draws the response curves by (29) and (30).

4. Learning procedure. Based on analytical solutions (27) and (28) and experimental data on the phase lag, one can, following [3], construct a learning procedure to compute the positive damping rates $\xi_{0i},\ \xi_{2i}$ as well as ξ_0 and ξ_1 (denote all of them as the vector $\boldsymbol{\xi} \ge 0$).

Appropriate measurements of the phase lag ψ for stable swirling were done in [5]. These measurements were employed in [7] to show that assuming the damping rates ($=$ viscous boundary layer thickness) of the order $O\big(\epsilon^{2/3}\big)$ (which means $\xi_{0i} = \xi_{2i} = 0$ and $\xi_1 = 0$ when neglecting the $o(\epsilon)$-order terms in the modal system) makes it impossible to fit, even qualitatively, the measured phase-lag data within the framework of the Narimanov – Moiseev-type modal theory.

Assume the given experimental measurements of the phase lag with the same liquid depth but, possibly, different forcing amplitudes, ϵ_x, are presented by the pairs

$$(\sigma/\sigma_{11} = s_j(\epsilon_x),\ \psi = p_j(\epsilon_x)), \quad j = 1, \ldots, N, \tag{31}$$

and consider the theoretical phase-lag branch

$$\left([\sigma/\sigma_{11}]_{\left[B,\alpha^{[*]}_{[B]}\right]}(\epsilon_x,\boldsymbol{\xi}) = \left(\Lambda_{\left[B,\alpha^{[*]}_{[B]}\right]}(\epsilon_x,\boldsymbol{\xi})+1\right)^{-1/2}, \psi_{\left[B,\alpha^{[*]}_{[B]}\right]}(\epsilon_x,\boldsymbol{\xi})\right) \tag{32}$$

as a function of $0 < B \le B_{\max}$.

Similar to [3], we also introduce the distance function between the measured (31) and theoretical (32) phase-lag branches as follows:

$$D(\epsilon_x,\boldsymbol{\xi},j) = \sqrt{\min_{0\le B\le B_{\max}} \left(s_j(\epsilon_x) - [\sigma/\sigma_{11}]_{\left[B,\alpha^{[*]}_{[B]}\right]}(\epsilon_x,\boldsymbol{\xi})\right)^2 + \left(p_j(\epsilon_x) - \psi_{\left[B,\alpha^{[*]}_{[B]}\right]}(\epsilon_x,\boldsymbol{\xi})\right)^2} \tag{33}$$

which is determined for each fixed experimental point j with the excitation amplitude ϵ_x and the trial vector $\boldsymbol{\xi} \ge 0$ of the damping rates. The integral (summarised) distance reads as

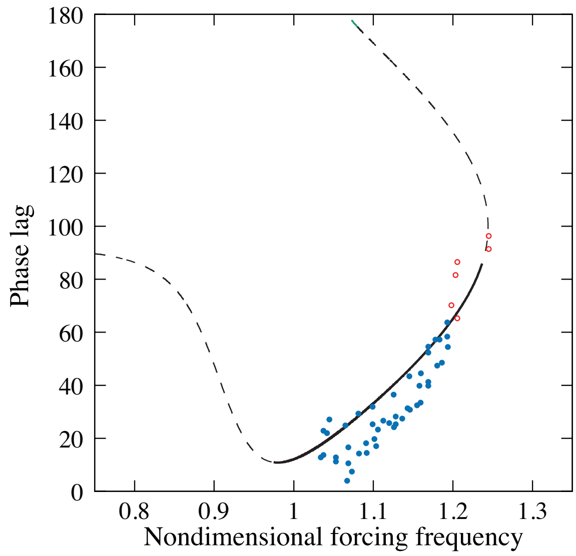
$$C(\boldsymbol{\xi}) = \sum_{j=1}^{N}\sum_{\epsilon_x} D(\epsilon_x,\boldsymbol{\xi},j). \tag{34}$$

It appears as the cost function in the learning procedure. Minimisation of the cost function (34) can be done by the gradient descent method.

5. Experimental measurements in [5]. To the authors best knowledge, the research paper [5] is a unique case when its authors measured the phase lag for steady-state resonant sloshing in an upright circular base tank exposed to a horizontal harmonic excitation. The measurements were done for the swirling-wave regime when the nondimensional mean liquid depth $h = 1.5$ and the excitation amplitude $\eta_{2a} = 0.045$. These were already employed in [7, 18] to investigate applicability of the Narimanov – Moiseev-type theory with $\xi_{11} \neq 0$ but $\xi_1 = \xi_{0k} = \eta_{2k} = 0$, $k = 1, \ldots, N_r$, which corresponds to the $O\left(\epsilon^{2/3}\right)$-order viscous boundary layer thickness. Similar failure was reported in [16] which used analogous assumptions to the Narimanov – Moiseev-type modal system [19] on sloshing in a two-dimensional rectangular tank. The present authors [3] showed that the latter modal system effectively fits the experimental data from [16] with viscous damping terms associated with the $O\left(\epsilon^{1/3}\right)$-asymptotics for the viscous boundary layer.

Fig. 1 demonstrates results of the developed (machine) learning procedure resulted from feature data on the phase lag in [5]. The left panel is drown when the learning procedure adopts the whole set of measurements including those where experimental observations discovered severe wave breakings with the free-surface fragmentation. The computed damping coefficients are $\xi_0 = 1.019368961\cdot10^{-4}$, $\xi_1 = 2.259451\cdot10^{-2}$, $\xi_{01} = 2.6424261\cdot10^{-2}$, $\xi_{02} = 6.4908885462\cdot10^{-3}$, $\xi_{03} = 0.21416321$, and $\xi_{21} = 0.150823783$, $\xi_{22} = 7.008502736 \cdot 10^{-3}$, $\xi_{23} = 7.00062716 \cdot 10^{-3}$. The right panel in Fig. 1 compares theory and experiments when feature data corresponding to severe breaking waves in [5] are excluded. The computed damping coefficients are $\xi_0 = 0.0$, $\xi_1 = 8.393626 \cdot 10^{-3}$, $\xi_{01} = 2.74034543 \cdot 10^{-2}$, $\xi_{02} = 8.7304159 \cdot 10^{-3}$, $\xi_{03} = 0.2148668$, and $\xi_{21} = 0.15094845$, $\xi_{22} = 7.138907 \cdot 10^{-3}$, $\xi_{23} = 7.31498666 \cdot 10^{-3}$.

The computed damping coefficients show that damping of higher natural sloshing modes plays primary role to fit experimental feature data. The damping rates ξ_{0i} and ξ_{2i} weakly change with



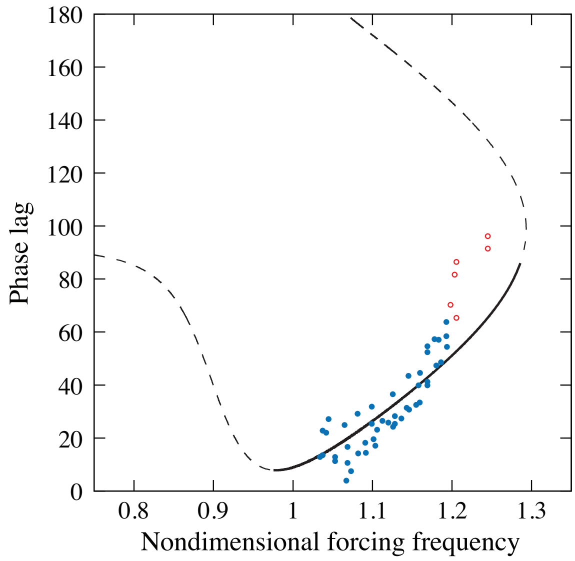


Fig. 1. Theoretical and experimental phase lag ψ (vertical axis, grad) versus the nondimensional forcing frequency σ/σ_{11} (horizontal axis) for longitudinal tank excitations with the nondimensional forcing amplitude $\eta_{2a} = 0.045$ and $h = 1.5$. Steady-state swirling waves. The measured values (circles) are taken from [5]. The theoretical response curves are marked by lines: the solid lines correspond to stable steady-state swirling (according to [7]) and the dashed lines imply instable waves. The empty (red) circles are experimental data for which [5] reports severe wave breaking with fragmentation of the free surface that can significantly affect, as discussed in [3], the measured data. The solid (deep blue) circles mark measurements done with stable swirling wave which is also accompanied by the breaking wave phenomenon but not much severe. The left panel demonstrates the results when all the measurement from [5] are adopted as feature data but the right panel was drown based on feature data which are marked by the solid circles.

choosing full or limited (trusted) feature data set. The linear damping of the lowest natural sloshing modes (generalised coordinates $p_{11}(t)$ and $r_{11}(t)$) can be practically neglected. But coefficient at the nonlinear damping term, ξ_1, is not zero and determines the theoretical branching behaviour versus the trusted (feature) experimental data.

6. Conclusions. In [7, 18], the Narimanov – Moiseev-type modal theory was applied to fit the experimental data on the phase lag measured for the resonant steady-state swirling wave in [5]. These attempts followed rather typical assumption that viscous damping in the hydrodynamic system can mainly be associated with energy dissipation of the dominant (lowest-order) generalised hydrodynamic coordinate in the adopted modal theory. The same failure was reported in [16] but for resonant sloshing in a rectangular tank. Solution on how to fix the latter problem was proposed in [3] where the present authors assumed that viscous damping is of strongly nonlinear character. Physically and asymptotically, this means that the viscous damping terms are of a lower asymptotic order. The present paper demonstrates that this idea (approach) is expandable onto the case in [5, 7, 18]. The comparative analysis states problem of whether all measurements in nonlinear sloshing can be adopted as feature data for machine learning of nonlinear modal equations.

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On behalf of all authors, the corresponding author states that there is no conflict of interest. All necessary data are included into the paper. Authors contribution: A. O. Miliaev — investigation (equal); methodology (equal); computations (equal); writing, original draft (equal); A. N. Timokha — conceptualization; investigation (equal); methodology (equal); computations (equal); writing, original draft (equal); supervision; writing, review and editing (equal). The second author acknowledges financial support of a grant from the Simons Foundation (1290607, A.T.) and the project 0123U100853 of the National Academy of Sciences of Ukraine.

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