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ON THE GEOMETRIC PROPERTIES OF SERIES IN SYSTEMS OF FUNCTIONS

ПРО ГЕОМЕТРИЧНІ ВЛАСТИВОСТІ РЯДІВ ЗА СИСТЕМОЮ ФУНКЦІЙ

Myroslav Sheremeta

Ivan Franko National University of Lviv
Universytetska St., 1, Lviv, 79000, Ukraine
e-mail: m.m.sheremeta@gmail.com

Let $f(z)=\sum_{k=1}^{\infty} f_k z^k$ be an entire transcendental function, let (λ_n) be a sequence of positive numbers increasing to $+\infty$, and let the series $A(z)=\sum_{n=1}^{\infty} a_n f(\lambda_n z)$ be regularly convergent in $\mathbb{D}=\{z\colon |z|<1\}$. The starlikeness and convexity of the function A are studied. For example, if $\sum_{n=1}^{\infty} \lambda_n^{-\tau}=T<+\infty$, $\ln|a_n|\le -e\lambda_n$, and $T\sum_{k=2}^{\infty} k|f_k|(k+\tau)^{k+\tau}\le \left|f_1\sum_{n=1}^{\infty} a_n\lambda_n\right|$, then the function A is starlike. It is proved that, under certain conditions on the parameters, the differential equation $z^2w''+(\beta_0z^2+\beta_1z)w'+(\gamma_0z^2+\gamma_1z+\gamma_2)w=0$ has an entire solution A that is starlike or convex in $\mathbb{D}$.

Нехай $f(z)=\sum_{k=1}^{\infty} f_k z^k$ — ціла трансцендентна функція, (λ_n) — зростаюча до $+\infty$ послідовність додатних чисел і ряд $A(z)=\sum_{n=1}^{\infty} a_n f(\lambda_n z)$ регулярно збіжний у $\mathbb{D}=\{z\colon |z|<1\}$. Вивчено зірковість і опуклість функції A. Наприклад, якщо $\sum_{n=1}^{\infty} \lambda_n^{-\tau}=T<+\infty$, $\ln|a_n|\le -e\lambda_n$ і $T\sum_{k=2}^{\infty} k|f_k|(k+\tau)^{k+\tau}\le \left|f_1\sum_{n=1}^{\infty} a_n\lambda_n\right|$, то функція A є зірковою. Доведено, що за певних умов на параметри диференціальне рівняння $z^2w''+(\beta_0z^2+\beta_1z)w'+(\gamma_0z^2+\gamma_1z+\gamma_2)w=0$ має цілий розв'язок A, який є зірковим або опуклим у $\mathbb{D}$.

1. Introduction. Let S be a class of functions

$$g(z)=z+\sum_{k=2}^{\infty} g_k z^k \tag{1}$$

analytic univalent in $\mathbb{D}=\{z\colon |z|<1\}$. The function $g\in S$ is said to be starlike if $g(\mathbb{D})$ is starlike domain concerning of the origin. It is well known [1, p. 202] that the condition $\operatorname{Re}\{zg'(z)/g(z)\}>0$, $z\in\mathbb{D}$, is necessary and sufficient for the starlikeness of g. A. W. Goodman [2] pointed out the conditions on the coefficients g_n under which function (1) is starlike. The concept of the starlikeness of function $g\in S$ got the series of generalizations. I. S. Jack [3] studied starlike functions of order $\alpha\in[0,1)$, i.e., such functions $g\in S$, for which $\operatorname{Re}\{zg'(z)/g(z)\}>\alpha$, $z\in\mathbb{D}$. V. P. Gupta [4] introduced the concept of starlike function of order $\alpha\in[0,1)$ and type $\beta\in(0,1]$. A function $g\in S$ is so named for that $|zg'(z)/g(z)-1|<\beta|zg'(z)/g(z)+1-2\alpha|$ for all $z\in\mathbb{D}$.

An analytic univalent in $\mathbb{D}=\{z\colon |z|<1\}$ function

$$\phi(z)=\sum_{n=0}^{\infty}\phi_n z^n \tag{2}$$

is said to be convex if $\phi(\mathbb{D})$ is a convex domain. It is clear that function (2) is convex if and only if function (1) with $g_n=\phi_n/\phi_1$ is convex. Also it is well known [1, p. 203] that the condition $\mathrm{Re}\,\{1+z\phi''(z)/\phi'(z)\}>0,\ z\in\mathbb{D}$, is necessary and sufficient for the convexity of φ. Therefore, function $g\in S$ is convex if and only if $\mathrm{Re}\,\{1+zg''(z)/g'(z)\}>0,\ z\in\mathbb{D}$, i.e., the function $g_1(z)=zg'(z)$ is starlike. By virtue of this remark, the function (1) is called [2] convex of order $\alpha\in[0,1)$ and type $\beta\in(0,1]$ if $|zg''(z)/g'(z)|<\beta|zg''(z)/g'(z)+2(1-\alpha)|$ for all $z\in\mathbb{D}$.

Let $f(z)=\sum_{k=0}^{\infty}f_k z^k$ be an entire transcendental function, (λ_n) be a sequence of positive numbers increasing to $+\infty$ and the series

$$A(z)=\sum_{n=1}^{\infty}a_n f(\lambda_n z) \tag{3}$$

regularly converges in $\{z\colon |z|<R[A]\}$, i.e., for all $r\in[0,R[A])$

$$\sum_{n=1}^{\infty}|a_n|M_f(r\lambda_n)<+\infty,\qquad M_f(r)=\max\{|f(z)|\colon |z|=r\}. \tag{4}$$

In the proposed note, we will investigate the conditions, under which function (3) is starlike, convex or close-to-convex. The results obtained are applicable to the study of the properties of solutions some differential equation.

2. Preliminary results. At first, we remark that if $R[A]\ge 1$, then in view of (4), for each $r<R[A]$, we have $|a_n|M_f(r\lambda_n)\le 1$ for $n\ge n_0(r)$, i.e., $\dfrac{1}{\lambda_n}M_f^{-1}\left(\dfrac{1}{|a_n|}\right)\ge r$, whence in view of the arbitrariness of r we get

$$\varliminf_{n\to\infty}\frac{1}{\lambda_n}M_f^{-1}\left(\frac{1}{|a_n|}\right)\ge R[A]\ge 1. \tag{5}$$

Consider a Dirichlet series

$$D(\sigma)=\sum_{n=1}^{\infty}|a_n|\lambda_n^{\sigma}=\sum_{n=1}^{\infty}|a_n|\exp\{\mu_n\sigma\},\quad \mu_n=\ln\lambda_n. \tag{6}$$

From (5) it follows that $1/|a_n|\ge M_f(c\lambda_n)$ for some $c>0$ and all n. Therefore,

$$\varliminf_{n\to\infty}\frac{1}{\mu_n}\ln\frac{1}{|a_n|}\ge\varliminf_{n\to\infty}\frac{\ln M_f(c\lambda_n)}{\ln\lambda_n}=+\infty,$$

because the function f is transcendental. Therefore, if $\ln n=O(\mu_n)$ as $n\to\infty$, then series (6) converges for all σ.

Let $\mu(\sigma)=\max\{|a_n|\exp\{\mu_n\sigma\}\colon n\ge 1\}$ be the maximal term of series (6). Denote by Ω a class of positive unbounded on $(-\infty,+\infty)$ functions Φ such that the derivative Φ' is positive, continuously differentiable, and increasing to $+\infty$ on $(-\infty,+\infty)$. Let φ be the function inverse to Φ' and $\Psi(x) = x - \Phi(x)/\Phi'(x)$ be the function associated with Φ in the sense of Newton. Then [5] in order that $\ln \mu(\sigma) \le \Phi(\sigma) \in \Omega$ for all σ it is necessary and sufficient that $\ln |a_n| \le -\mu_n \Psi(\varphi(\mu_n))$ for all n. Therefore, if we put $\mu_n = \ln \lambda_n$ and $\sigma = k + \tau$, then for hence we obtain

$$\max\left\{|a_n|\lambda_n^{k+\tau} : n \ge 2\right\} \le e^{\Phi(k+\tau)} \tag{7}$$

provided $\lambda_n \ge 1$ and $\ln |a_n| \le -\ln \lambda_n \Psi(\varphi(\ln \lambda_n))$ for all $n \ge 2$.

3. Starlikeness and convexity. We need the following lemma [2] (see also [6, p. 9]).

Lemma 1. *If* $\sum_{k=2}^{\infty} k|g_k| \le 1$, *then function (1) is starlike, and if* $\sum_{k=2}^{\infty} k^2|g_k| \le 1$, *then function (1) is convex.*

We will further assume that

$$f(z) = f_1 z + \sum_{k=2}^{\infty} f_k z^n \tag{8}$$

and $f_1 \sum_{n=1}^{\infty} a_n \lambda_n \ne 0$. Also we assume that the sequence (λ_n) has a finite convergence index, i.e., there exist positive numbers τ and T such that

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^{\tau}} = T < +\infty. \tag{9}$$

Hence, it follows that $\ln n = O(\ln \lambda_n)$ as $n \to \infty$.

Theorem 1. *Let* $R(A) \ge 1,\ \Phi \in \Omega,\ \lambda_n \ge 1,\ \ln |a_n| \le -\ln \lambda_n \Psi(\varphi(\ln \lambda_n))$ *for* $n \ge 1$ *and* $f_1 \sum_{n=1}^{\infty} a_n \lambda_n \ne 0$. *Suppose that (8) and (9) hold. Then:*

(i) *if* $T \sum_{k=2}^{\infty} k|f_k| e^{\Phi(k+\tau)} \le \left|f_1 \sum_{n=1}^{\infty} a_n \lambda_n\right|$, *then function (3) is starlike in* $\mathbb{D}$*;*

(ii) *if* $T \sum_{k=2}^{\infty} k^2|f_k| e^{\Phi(k+\tau)} \le \left|f_1 \sum_{n=1}^{\infty} a_n \lambda_n\right|$, *then function (3) is convex in* $\mathbb{D}$.

Proof. Since

$$\begin{aligned} A(z) &= \sum_{n=1}^{\infty} a_n \sum_{k=1}^{\infty} f_k \lambda_n^k z^k = \sum_{k=0}^{\infty} f_k \left(\sum_{n=1}^{\infty} a_n \lambda_n^k\right) z^k \\ &= f_1 \sum_{n=1}^{\infty} a_n \lambda_n z + \sum_{k=2}^{\infty} f_k \left(\sum_{n=1}^{\infty} a_n \lambda_n^k\right) z^k \\ &= f_1 \sum_{n=1}^{\infty} a_n \lambda_n \left(z + \sum_{k=2}^{\infty} \frac{f_k \sum_{n=1}^{\infty} a_n \lambda_n^k}{f_1 \sum_{n=1}^{\infty} a_n \lambda_n} z^k\right), \end{aligned} \tag{10}$$

whence it follows that function (3) is starlike (convex) if and only if function (1) with

$$g_k = \frac{f_k \sum_{n=1}^{\infty} a_n \lambda_n^k}{f_1 \sum_{n=1}^{\infty} a_n \lambda_n}, \quad k \ge 2, \tag{11}$$

is starlike (convex). Therefore, if

$$\sum_{k=2}^{\infty} k \left| f_k \sum_{n=1}^{\infty} a_n \lambda_n^k \right| \le \left| f_1 \sum_{n=1}^{\infty} a_n \lambda_n \right|, \tag{12}$$

then by Lemma 1 function (3) is starlike, and if

$$\sum_{k=2}^{\infty} k^2 \left| f_k \sum_{n=1}^{\infty} a_n \lambda_n^k \right| \le \left| f_1 \sum_{n=1}^{\infty} a_n \lambda_n \right|, \tag{13}$$

then function (3) is convex in $\mathbb{D}$.

By condition (9) in view of (7) we have

$$\sum_{n=1}^{\infty} |a_n| \lambda_n^k = \sum_{n=1}^{\infty} \frac{|a_n| \lambda_n^{k+\tau}}{\lambda_n^{\tau}} \le T \max\{|a_n| \lambda_n^{k+\tau} : n \ge 1\} \le T e^{\Phi(k+\tau)},$$

i.e., (12) holds provided $T \sum_{k=2}^{\infty} k|f_k| e^{\Phi(k+\tau)} \le \left| f_1 \sum_{n=1}^{\infty} a_n \lambda_n \right|$ and (13) holds provided $T \sum_{k=2}^{\infty} k|f_k| e^{\Phi(k+\tau)} \le \left| f_1 \sum_{n=1}^{\infty} a_n \lambda_n \right|$.

Theorem 1 is proved.

Let us consider several consequences of Proposition 1. At first, let $\Phi(x) = e^x$ for $x > 0$. Then $\Psi(x) = x - 1,\ \varphi(x) = \ln x$, and $x\Psi(\varphi(x)) = x \ln(x/e)$. Therefore, we get the following statement.

Corollary 1. *Let (8) and (9) hold. Suppose that $\lambda_n > 1$ and $\ln|a_n| \le -\ln \lambda_n \ln\left(\frac{\ln \lambda_n}{e}\right)$ for $n \ge 1$. If $T \sum_{k=2}^{\infty} k|f_k| \exp\{e^{k+\tau}\} \le \left| f_1 \sum_{n=1}^{\infty} a_n \lambda_n \right|$, then function (3) is starlike, and if $T \sum_{k=2}^{\infty} k^2|f_k| \exp\{e^{k+\tau}\} \le \left| f_1 \sum_{n=1}^{\infty} a_n \lambda_n \right|$, then function (3) is convex.*

Now let $\Phi(x) = x^p$ for $x > 1$, where $p > 1$. Then $\Psi(x) = \frac{p-1}{p} x,\ \varphi(x) = \left(\frac{x}{p}\right)^{1/(p-1)}$, and $x\Psi(\varphi(x)) = (p-1)\left(\frac{x}{p}\right)^{p/(p-1)}$. Therefore, we get the following statement.

Corollary 2. *Let (8) and (9) hold. Suppose that $\lambda_n > 1$ and $\ln|a_n| \le -(p-1)\left(\frac{\ln \lambda_n}{p}\right)^{p/(p-1)}$ for $n \ge 1$. If $T \sum_{k=2}^{\infty} k|f_k| \exp\{(k+\omega)^p\} \le \left| f_1 \sum_{n=1}^{\infty} a_n \lambda_n \right|$, then function (3) is starlike, and if $T \sum_{k=2}^{\infty} k^2|f_k| \exp\{(k+\tau)^p\} \le \left| f_1 \sum_{n=1}^{\infty} a_n \lambda_n \right|$, then function (3) is convex.*

Finally, consider the most interesting case when $\Phi(x) = x \ln x$ for $x > e$. Then $\varphi(x) = e^{x-1}$ and $x\Psi(\varphi(x)) = x\varphi(x) - \Phi(\varphi(x)) = e^{x-1}$. Therefore, we get the following statement.

Corollary 3. *Let (8) and (9) hold. Suppose that $\lambda_n \ge 1$ and $\ln|a_n| \le -e\lambda_n$ for $n \ge 1$. If $T \sum_{k=2}^{\infty} k|f_k|(k+\tau)^{k+\tau} \le \left| f_1 \sum_{n=1}^{\infty} a_n \lambda_n \right|$, then function (3) is starlike, and if $T \sum_{k=2}^{\infty} k^2|f_k|(k+\tau)^{k+\tau} \le \left| f_1 \sum_{n=1}^{\infty} a_n \lambda_n \right|$, then function (3) is convex.*

Remark 1. In [2] it is proved that if $\alpha \in [0, 1),\ \beta \in (0, 1]$, and

$$\sum_{k=2}^{\infty} \{(1+\beta)k - 2\beta\alpha - (1-\beta)\}|g_k| \le 2\beta(1-\alpha),$$

then function (1) is starlike of order α and type β, and if

$$\sum_{k=2}^{\infty} k\big\{(1+\beta)k - 2\beta\alpha - (1-\beta)\big\}|g_k| \le 2\beta(1-\alpha),$$

then function (1) is convex of order α and type β. Therefore, as above, we get the following statement.

Proposition 1. *Let $\alpha \in [0,1),\ \beta \in (0,1]$, and the conditions of Theorem 1 are fulfilled. If*

$$T\sum_{k=2}^{\infty}\big\{(1+\beta)k-2\beta\alpha-(1-\beta)\big\}|f_k|e^{\Phi(k+\tau)} \le 2\beta(1-\alpha)\left|f_1\sum_{n=1}^{\infty}a_n\lambda_n\right|,$$

then function (3) is starlike of order α and type β, and if

$$T\sum_{k=2}^{\infty}k\big\{(1+\beta)k-2\beta\alpha-(1-\beta)\big\}|f_k|e^{\Phi(k+\tau)} \le 2\beta(1-\alpha)\left|f_1\sum_{n=1}^{\infty}a_n\lambda_n\right|,$$

then function (3) is convex of order α and type β.

4. Close-to-convexity. By W. Kaplan [7], function (2) is said to be close-to-convex in $\mathbb{D}$ if there exists a convex in $\mathbb{D}$ function F such that $\mathrm{Re}\,(\phi'(z)/F'(z)) > 0,\ z \in \mathbb{D}$. Close-to-convex function ϕ has a characteristic property that the complement G of the domain $\phi(\mathbb{D})$ can be filled with rays L which go from ∂G and lie in G. Every close-to-convex in $\mathbb{D}$ function f is univalent in $\mathbb{D}$ and, therefore, $f'(0) \ne 0$. Hence it follows that the function ϕ is close-to-convex in $\mathbb{D}$ if and only if the function (1) with $g_n = \phi_n/\phi_1$ is close-to-convex in $\mathbb{D}$.

For function (1) the following Alexander criterion is valid [8] (see also [6, p. 11]): *if*

$$1 \ge 2g_2 \ge \ldots \ge kg_k \ge (k+1)g_{k+1} \ge \ldots > 0, \tag{14}$$

then function (1) is close-to-convex in $\mathbb{D}$. For coefficients (11) condition (14) is equivalent to condition

$$1 \ge \frac{2f_2\sum_{n=1}^{\infty}a_n\lambda_n^2}{f_1\sum_{n=1}^{\infty}a_n\lambda_n} \ge \ldots \ge \frac{kf_k\sum_{n=1}^{\infty}a_n\lambda_n^k}{f_1\sum_{n=1}^{\infty}a_n\lambda_n} \ge \frac{(k+1)f_{k+1}\sum_{n=1}^{\infty}a_n\lambda_n^{k+1}}{f_1\sum_{n=1}^{\infty}a_n\lambda_n} \ge \ldots > 0.$$

Therefore, the following statement is true.

Proposition 2. *Let (8) holds and either all coefficients f_k and a_n are positive or all coefficients f_k and a_n are negative. If*

$$f_1\sum_{n=1}^{\infty}a_n\lambda_n \ge 2f_2\sum_{n=1}^{\infty}a_n\lambda_n^2 \ge \ldots \ge kf_k\sum_{n=1}^{\infty}a_n\lambda_n^k \ge (k+1)f_{k+1}\sum_{n=1}^{\infty}a_n\lambda_n^{k+1} \ge \ldots > 0,$$

then function (3) close-to-convex in $\mathbb{D}$.

Choosing the function F appropriately, one can obtain other sufficient conditions for the close-to-convexity of the function g. For example, if $F(z) = \ln\dfrac{1}{1-z}$, then, as in [6, pp. 11, 12], we have

$$\frac{g'(z)}{F'(z)} = 1 + \sum_{k=1}^{\infty}((k+1)g_{k+1} - kg_k)z^k.$$

Suppose that all $g_k > 0,\ kg_k \nearrow \xi < 2$ as $k \to \infty$ and put $F_n(z) = 1 + \sum_{k=1}^{n}((k+1)g_{k+1} - kg_k)z^k$. Then, for all $z \in \mathbb{D}$,

$$\mathrm{Re}\,F_n(z) \ge 1 - \left|\sum_{k=1}^{n}((k+1)g_{k+1} - kg_k)z^k\right|$$

$$\geq 1-\sum_{k=1}^{n}((k+1)g_{k+1}-kg_k)=2-(n+1)g_{n+1}\geq 2-\xi>0.$$

Since $\dfrac{g'(z)}{F'(z)}=\lim_{n\to\infty}F_n(z)$, from hence it follows that function (1) is close-to-convex and, thus, the following statement is proved.

Proposition 3. *Let (8) holds and either all coefficients f_k and a_n are positive or all coefficients f_k and a_n are negative. If $\left(kf_k\sum_{n=1}^{\infty}a_n\lambda_n^k\right)/\left(f_1\sum_{n=1}^{\infty}a_n\lambda_n\right)\nearrow\xi<2$ as $k\to\infty$, then function (3) close-to-convex in $\mathbb{D}$.*

5. Shah's differential equation. S. M. Shah [9] indicated conditions on real parameters β_0, $\beta_1,\ \gamma_0,\ \gamma_1,\ \gamma_2$ of the differential equation

$$z^2w''+(\beta_0z^2+\beta_1z)w'+(\gamma_0z^2+\gamma_1z+\gamma_2)w=0, \tag{15}$$

under which there exists an entire transcendental solution $g(z)=\sum_{k=0}^{\infty}g_kz^k$ that together with its derivatives are close-to-convex in $\mathbb{D}$. It easy to show that g is a solution of (15) if and only if $\gamma_2g_0=0,\ (\beta_1+\gamma_2)g_1+\gamma_1g_0=0$ and

$$g_k=-\frac{\beta_0(k-1)+\gamma_1}{k(k+\beta_1-1)+\gamma_2}g_{k-1}-\frac{\gamma_0}{k(k+\beta_1-1)+\gamma_2}g_{k-2},\quad k\geq 2. \tag{16}$$

If either $\gamma_0=0$ or $\beta_0=\gamma_1=0$, the two-term recurrent formula turns into a one-term recurrent formula and these cases are studied in [9]. In the general case the existence of solutions that together with its derivatives are close-to-convex in $\mathbb{D}$ was studied in [10, 11]. The convexity of such solutions was studied in [12].

The question of the existence of a solution (3) to the differential equation (15) is natural and can be solved in the general case of a two-term recurrent formula for coefficients, but we will limit ourselves to the case of a special one-term recurrent formula.

We choose $\beta_0=\beta_1=\gamma_0=\gamma_2=0$. Then for $z\neq 0$ from (15) we obtain $zw''+\gamma_1w=0$ and (16) implies $g_k=-\dfrac{\gamma_1}{k(k-1)}g_{k-1}$ for $k\geq 2$. In view of the equalities $\gamma_2g_0=0$ and $(\beta_1+\gamma_2)g_1+\gamma_1g_0=0$, the coefficients g_0 and g_1 may be arbitrary. Choosing $g_0=0,\ g_1=1$, and g_k by using the formula (11), hence we get that the function A is a solution of the differential equation $zw''+\gamma_1w=0$ if and only if

$$f_k\sum_{n=1}^{\infty}a_n\lambda_n^k=-\frac{\gamma_1}{k(k-1)}f_{k-1}\sum_{n=1}^{\infty}a_n\lambda_n^{k-1},\quad k\geq 2. \tag{17}$$

By using this formula we prove the following theorem.

Theorem 2. *Let $\beta_0=\beta_1=\gamma_0=\gamma_2=0$ and $\gamma_1\neq 0$. Then Shah's differential equation (15) has an integer solution (10) with coefficients satisfying condition (17) and such that if $|\gamma_1|\leq 4/5$, then function A is starlike and if $|\gamma_1|\leq 8/19$, then function A is convex in $\mathbb{D}$.*

Proof. If $|\gamma_1|\leq 4/5$, then from (17) we have

$$\sum_{k=2}^{\infty}k\left|f_k\sum_{n=1}^{\infty}a_n\lambda_n^k\right|\leq\sum_{k=2}^{\infty}k\frac{|\gamma_1|}{k(k-1)}\left|f_{k-1}\sum_{n=1}^{\infty}a_n\lambda_n^{k-1}\right|$$

$$= |\gamma_1|\left|f_1\sum_{n=1}^{\infty}a_n\lambda_n\right| + |\gamma_1|\sum_{k=2}^{\infty}\frac{1}{k}\left|f_k\sum_{n=1}^{\infty}a_n\lambda_n^k\right|$$

$$= |\gamma_1|\left|f_1\sum_{n=1}^{\infty}a_n\lambda_n\right| + |\gamma_1|\sum_{k=2}^{\infty}\frac{1}{k^2}k\left|f_k\sum_{n=1}^{\infty}a_n\lambda_n^k\right|$$

$$\le |\gamma_1|\left|f_1\sum_{n=1}^{\infty}a_n\lambda_n\right| + \frac{|\gamma_1|}{4}\sum_{k=2}^{\infty}k\left|f_k\sum_{n=1}^{\infty}a_n\lambda_n^k\right|$$

and, thus,

$$\left(1-\frac{|\gamma_1|}{4}\right)\sum_{k=2}^{\infty}k\left|f_k\sum_{n=1}^{\infty}a_n\lambda_n^k\right| \le |\gamma_1|\left|f_1\sum_{n=1}^{\infty}a_n\lambda_n\right|,$$

whence

$$\sum_{k=2}^{\infty}k\left|f_k\sum_{n=1}^{\infty}a_n\lambda_n^k\right| \le \frac{4|\gamma_1|}{4-|\gamma_1|}\left|f_1\sum_{n=1}^{\infty}a_n\lambda_n\right| \le \left|f_1\sum_{n=1}^{\infty}a_n\lambda_n\right|,$$

because $|\gamma_1| \le 4/5$, i.e., (12) holds and function (10) is starlike.

If $|\gamma_1| \le 8/19$, then similarly

$$\sum_{k=2}^{\infty}k^2\left|f_k\sum_{n=1}^{\infty}a_n\lambda_n^k\right| \le 2|\gamma_1|\left|f_1\sum_{n=1}^{\infty}a_n\lambda_n\right| + \frac{3|\gamma_1|}{8}\sum_{k=2}^{\infty}k^2\left|f_k\sum_{n=1}^{\infty}a_n\lambda_n^k\right|,$$

whence

$$\sum_{k=2}^{\infty}k^2\left|f_k\sum_{n=1}^{\infty}a_n\lambda_n^k\right| \le \frac{16|\gamma_1|}{8-3|\gamma_1|}\left|f_1\sum_{n=1}^{\infty}a_n\lambda_n\right| \le \left|f_1\sum_{n=1}^{\infty}a_n\lambda_n\right|,$$

i.e., (13) holds and function (10) is convex.

Finally, (17) implies

$$\ln\frac{1}{\left|f_k\sum\limits_{n=1}^{\infty}a_n\lambda_n^k\right|} = \ln\frac{1}{\left|f_{k-1}\sum_{n=1}^{\infty}a_n\lambda_n^{k-1}\right|} + \ln\frac{k(k-1)}{|\gamma_1|}$$

$$= \ln\frac{1}{\left|f_{k-2}\sum_{n=1}^{\infty}a_n\lambda_n^{k-2}\right|} + \ln\frac{k(k-1)}{|\gamma_1|} + \ln\frac{(k-1)(k-2)}{|\gamma_1|} = \ldots$$

$$= \ln\frac{1}{\left|f_1\sum_{n=1}^{\infty}a_n\lambda_n\right|} + \sum_{j=2}^{k}\ln\frac{j(j-1)}{|\gamma_1|},$$

whence it follows that

$$\frac{1}{k}\ln\frac{1}{\left|f_k\sum_{n=1}^{\infty}a_n\lambda_n^k\right|} \to +\infty, \quad k\to\infty,$$

i.e., function (10) is entire.

The proof of Theorem 2 is complete.

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