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TWO-POINT BOUNDARY-VALUE PROBLEM FOR LINEAR STOCHASTIC DIFFERENTIAL EQUATIONS

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We present the extension of the two-point boundary problem in the case where the problem doesn't always have a unique solution or the condition on boundary coefficients described in [*Linear stochastic differential equations with boundary conditions*, Probab. Th. Rel. Fields, **82**, 489–526 (1989)] is not satisfied.

Наведено розширення двоточкової крайової задачі у випадку, коли задача не завжди має єдиний розв'язок або коли умова на граничні коефіцієнти, описана в [*Linear stochastic differential equations with boundary conditions*, Probab. Th. Rel. Fields, **82**, 489 – 526 (1989)], не виконується.

This paper deals with methods of analysing of solutions of two-point boundary-value problem for systems of linear stochastic differential equations. The cases when the boundary-value problem is not always solved and doesn't always have a unique solution were investigated.

Boundary-value problem for linear stochastic differential equations were studied by many authors. Some of them are Nikoukhah, Willsky & Levy [1], Kwakernaak [2], Mashkov, Fedchenko and Prigarin [3], Alabert and Ferrante [4], Mashkov [5] used pseudo-inverse matrices theory in order to derive the sufficient conditions for the existence of solution stochastic system for Gaussian case and initial condition. Nualart and Pardoux [6] developed an extended stochastic calculus for generalized Stratonovich integral and for Skorohod integral, Ocone and Pardoux [7] used generalized Stratonovich integral for two-point boundary-value problem for non-Gaussian case. This study presents the extension of two-point boundary problem in the case when the problem doesn't always have a unique solution or when condition on boundary coefficients described in [7] is not hold. Results are based on the methods developed by Boichuk and Samoilenko [8]. As in Ocone and Pardoux [7] it is also considering generalized Stratonovich integral $\int_0^t f(s, W_s) \circ dW_t$ in order to work with functions that anticipate the driving Wiener process. Consequently, this allows us to build two-point boundary-value problem.

Let us give some definitions:

Definition 1. *The filtration generated stochastic process X is the family of σ-algebras $\{\mathcal{F}_t^X \mid t \geq 0\}$, where $\mathcal{F}_t^X = \sigma\Big(\bigcup_{s\leq t} \sigma(X_s)\Big)$. Each $\mathcal{F}_t^X$ is a σ-algebra, and if $s \leq t$, then $\mathcal{F}_s^X \subseteq \mathcal{F}_t^X$. A family of σ-algebras with this property is called filtration.*

Definition 2. *A stochastic process X is called adapted (non-anticipative) to filtration* $\{\mathcal{F}_t \mid t > 0\}$, *if, for all* $t \geq 0$, $X(t)$ *is* $\mathcal{F}_t$*-measurable. Value* $X(t)$ *can be determined by the "information available at time t".*

For all details about generalized Stratonovich integral and why it is suitable for two-point boundary-value problem we refer to [7].

Presentation of the problem. Let $W_t,\ t \in [0,1]$, denote a Brownian motion defined on the space $(\Omega, \mathcal{F}_t, P)$. $\mathcal{F}_t$ is the filtration of W_t. We study the solvability problem of the following equations on $[0,1]$:

$$dX_t = (AX_t + a(t))\,dt + \sum_{i=1}^{k}(B_iX_t + b_i(t)) \circ dW_t^i, \tag{1.1}$$

$$F_0X_0 + F_1X_1 = f, \tag{1.2}$$

where $A, B_1, \ldots, B_k, F_0, F_1$ are $(d \times d)$-dimensional matrices, f is a d-dimensional random vector defined on (Ω, F), $a, b_1, \ldots, b_k$ are d-dimensional processes.

In the case where $B_i = 0,\ b_i(t)$ is constant, for all i and $a(t) = 0$, then (1.1) is of Gaussian type, and it is not necessary to use the generalized Stratonovich integral since the stochastic integrals do not contain an anticipating term. Ocone and Pardoux [7] have studied (1.1), (1.2) for the case where $\operatorname{rank}(F_0 : F_1) = d$. We will introduce cases where this condition does not hold.

It is well known the derivation of solution of (1.1) (see [9] for integral Ito case). Let us write the homogeneous part of the equation (1.1):

$$d\Phi_t = A\Phi_t\,dt + B_i\Phi_t \circ dW_t^i,$$

$$\Phi_0 = I.$$

We find fundamental solution Φ_t, which is $(d \times d)$-dimensional matrix of the equation (1.1), Φ_t is adapted to the filtration $\mathcal{F}_t,\ t \in [0,1]$.

Define

$$\Phi(t,s) = \Phi_t\,\Phi_s^{-1}, \quad t,s \in [0,1].$$

By using the variation of constant formula, the solution of (1.1) is given by

$$X_t = \Phi(t,0)X_0 + \int_0^t \Phi(t,s) \circ dV_s, \tag{1.3}$$

where

$$V_t = \int_0^t a(s)\,ds + \sum_{i=1}^{k}\int_0^t b_i(s) \circ dW_s^i, \quad t \in [0,1],$$

and X_t is a d-dimensional unknown vector. Putting the solution of (1.1) into the boundary condition (1.2), we get the following equation where we find the unknown vector X_0:

$$[F_0 + F_1\Phi(1,0)]X_0 = f - F_1\int_0^1 \Phi(1,s) \circ dV_s. \tag{1.4}$$

If the matrix $[F_0 + F_1\Phi(1,0)]$ is invertible, then X_0 has a unique solution [7]. Denote $Q = [F_0 + F_1\Phi(1,0)]$ as a $(d \times d)$-dimensional matrix, and in the case where Q is not invertible, we will establish a solvability condition for (1.4) [8]:

$$P_{Q_l^*}\left\{ f - F_1 \int\limits_0^1 \Phi(1,s) \circ dV_s \right\} = 0,$$

where $l = d - \operatorname{rank}(Q)$ and $P_{Q_l^*}$ is matrix-projector onto the cokernel of matrix Q_l. Then

$$X_0 = P_{Q_r} c_r + Q^+ \left\{ f - F_1 \int\limits_0^1 \Phi(1,s) \circ \, dV_s \right\}, \quad r = d - \operatorname{rank}(Q),$$

where Q^+ is the Moore – Penrose pseudoinverse matrix and P_{Q_r} is a $(d \times r)$-dimensional matrix-projector onto the kernel of Q. Then, putting X_0 into equation (1.3) yields the final solution to problem (1.1), (1.2):

$$X_t = \Phi(t,0) \left\{ P_{Q_r} c_r + Q^+ \left[f - F_1 \int\limits_0^1 \Phi(1,s) \circ dV_s \right] \right\} + \int\limits_0^t \Phi(t,s) \circ dV_s.$$

Theorem. *The boundary-value problem (1.1), (1.2) is solvable if and only if f, $a(t)$, and $b(t)$ satisfy l linear independent conditions*

$$P_{Q_l^*}\left\{ f - F_1 \int\limits_0^1 \Phi(1,s) \circ dV_s \right\} = 0, \quad l = d - \operatorname{rank}(Q). \tag{1.5}$$

In this case, the boundary problem (1.1), (1.2) possesses an r-parameter $(r = d - \operatorname{rank}(Q)$, $\operatorname{rank}(Q) \leq d)$ family of solutions

$$X_t = \Phi(t,0) \left[P_{Q_r} c_r + Q^+ \left\{ f - F_1 \int\limits_0^1 \Phi(1,s) \circ dV_s \right\} \right] + \int\limits_0^t \Phi(t,s) \circ dV_s.$$

Thus, by using generalized Stratonovich integral and methods of analysis boundary-value problems described in [8], we can extend results of [7] for getting general solution of problem (1.1), (1.2).

Corollary 1. *If $\operatorname{rank}(Q) = d$, then the problem (1.1), (1.2) has a unique solution*

$$X_t = \Phi(t,0) \left\{ Q^{-1} \left\{ f - F_1 \int\limits_0^1 \Phi(1,s) \circ dV_s \right\} \right\} + \int\limits_0^t \Phi(t,s) \circ dV_s.$$

We arrive at the same as in [7].

Corollary 2. *If* $\operatorname{rank} Q = d_1,\ d_1 < d$, *then the problem (1.1), (1.2) is solvable if and only if* $f,\ a(t),\ b(t)$ *satisfy* l *linear independent conditions*

$$P_{Q_l^*}\left\{ f - F_1 \int\limits_0^1 \Phi(1,s) \circ dV_s \right\} = 0, \quad l = d - d_1.$$

And in this case (1.1), (1.2) possesses an r*-parameter* $(r = d - d_1)$ *family of solutions*

$$X_t = \Phi(t,0)\left\{ P_{Q_r} c_r + Q^+ \left\{ f - F_1 \int\limits_0^1 \Phi(1,s) \circ dV_s \right\}\right\} + \int\limits_0^t \Phi(t,s) \circ dV_s.$$

In order to check condition (1.5), numerical methods can be used.

Let us apply theorem to the following equations.

Example 1 [7]. Let us $B_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},\ B_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ a(t) = b(t) = 0,\ f = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ F_0 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, and $F_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

In this case $\operatorname{rank}(F_0 : F_1) = 2$,

$$\Phi(t,s) = \begin{pmatrix} e^{W_t^1 - W_s^1} & \alpha_t^s \\ 0 & e^{W_t^2 - W_s^2} \end{pmatrix},$$

where $\alpha_t^s = e^{W_t^1} \int_s^t e^{-W_u^1} e^{W_u^2 - W_s^2} dW_u^1$.

Then $Q = F_0 + F_1,\ \Phi(1,0) = \begin{pmatrix} 1 & 1 \\ 0 & e^{W_1^2} \end{pmatrix},\ \operatorname{rank} Q = 2$ (with $P1$).

So, $\exists! X_0 = Q^+ f = \begin{pmatrix} 1 - e^{-W_1^2} \\ e^{-W_1^2} \end{pmatrix}$ and

$$X_t = \Phi(t,0) Q^+ f = \begin{pmatrix} e^{W_t^1}\left(1 - e^{-W_1^2}\right) + \alpha_t^s e^{-W_1^2} \\ e^{W_t^2 - W_1^2} \end{pmatrix}.$$

Example 2. The same coefficients as in Example 1.1 except F_0 and F_1:

$$F_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad F_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

In this case $Q = \begin{pmatrix} 1 & e^{W_1^2} \\ 0 & 0 \end{pmatrix},\ \operatorname{rank}(Q) = 1,\ l = d - \operatorname{rank}\left(Q^*\right) = 1$. By using solvability condition, will check that

$$P_{Q_1^*} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0.$$

Firstly, the pseudo-inverse matrix is found:

$$Q^+ = \lim_{\varepsilon\to 0}(Q^*Q + \varepsilon I)^{-1}Q^*,$$

$$Q^+ = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon + \varepsilon e^{2W_1^2} + \varepsilon^2}\begin{pmatrix} \varepsilon & 0 \\ -e^{W_1^2} + (1+\varepsilon)e^{W_1^2} & 0 \end{pmatrix} = \frac{1}{1+e^{2W_1^2}}\begin{pmatrix} 1 & 0 \\ e^{W_1^2} & 0 \end{pmatrix}.$$

By using formula $P_{Q^*} = I - QQ^+$, we obtain

$$P_{Q^*} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$P_{Q_1^*} = \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

Checking solvability condition

$$P_{Q_1^*} f = 1 \neq 0.$$

Thus, the equation is unsolvable, but there is the pseudosolution

$$X_0^+ = Q^+ f, \quad \text{where} \quad f = (1,1)^T,$$

$$X_0^+ = \frac{1}{1+e^{2W_1^2}}\begin{pmatrix} 1 \\ e^{W_1^2} \end{pmatrix},$$

and the general solution is

$$X_t = \Phi(t,0)\big(P_{Q_r}c_r + X_0^+\big), \quad r = 1.$$

By using formula $P_Q = I - Q^+Q$, we get

$$P_Q = \frac{1}{1+e^{2W_1^2}}\begin{pmatrix} e^{2W_1^2} & -e^{W_1^2} \\ -e^{W_1^2} & 1 \end{pmatrix}$$

and

$$P_{Q_1} = \frac{1}{1+e^{2W_1^2}}\begin{pmatrix} -e^{W_1^2} \\ 1 \end{pmatrix},$$

$$X_t = \begin{pmatrix} e^{W_t^1} & \alpha_t^0 \\ 0 & e^{W_t^2} \end{pmatrix}\left(\frac{1}{1+e^{2W_1^2}}\begin{pmatrix} -e^{W_1^2} \\ 1 \end{pmatrix} c_1 + \frac{1}{1+e^{2W_1^2}}\begin{pmatrix} 1 \\ e^{W_1^2} \end{pmatrix}\right)$$

$$= \frac{1}{1+e^{2W_1^2}}\begin{pmatrix} -e^{W_1^2+W_t^1}c_1 + e^{W_t^1} + c_1\alpha_t^0 + \alpha_t^0 e^{W_1^2} \\ c_1 e^{W_t^2} + e^{W_1^2+W_t^2} \end{pmatrix}.$$

Example 3. Let

$$B_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad F_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad f = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad a(t) = b(t) = 0.$$

In this case, $\operatorname{rank}(F_0 : F_1) = 2$.

Since $B_1 B_2 = B_2 B_1$, hence

$$\Phi(t,s) = \exp\bigl(B_1\bigl(W_t^1 - W_s^1\bigr) + B_2\bigl(W_t^2 - W_s^2\bigr)\bigr)$$
$$= \exp\bigl(B_1\bigl(W_t^1 - W_s^1\bigr)\bigr)\exp\bigl(B_2\bigl(W_t^2 - W_s^2\bigr)\bigr),$$

$$\exp\bigl(B_1\bigl(W_t^1 - W_s^1\bigr)\bigr) = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \exp\left(\bigl(W_t^1 - W_s^1\bigr)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)\begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{(W_t^1 - W_s^1)} & -1 + e^{(W_t^1 - W_s^1)} & 0 \\ 0 & 1 & 0 \\ 0 & -1 + e^{(W_t^1 - W_s^1)} & e^{(W_t^1 - W_s^1)} \end{pmatrix},$$

$$\exp\bigl(B_2\bigl(W_t^2 - W_s^2\bigr)\bigr) = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \exp\left(\bigl(W_t^2 - W_s^2\bigr)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} e^{(W_t^2 - W_s^2)} & -1 + e^{(W_t^2 - W_s^2)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\Phi(t,s) = \begin{pmatrix} e^{(W_t^1 - W_s^1) + (W_t^2 - W_s^2)} & e^{(W_t^1 - W_s^1) + (W_t^2 - W_s^2)} + 2e^{(W_t^1 - W_s^1)} - 1 & 0 \\ 0 & 1 & 0 \\ 0 & e^{(W_t^1 - W_s^1)} - 1 & e^{(W_t^1 - W_s^1)} \end{pmatrix}$$

and

$$Q = F_0 + F_1\Phi(1,0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e^{W_1^1} & e^{W_1^1} \end{pmatrix},$$

$$Q^{+} = \lim_{\varepsilon\to 0}(Q^{*}Q+\varepsilon I)^{-1}Q^{*} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \dfrac{e^{-W_1^1}}{2} \\ 0 & 0 & \dfrac{e^{-W_1^1}}{2} \end{pmatrix},$$

$$P_{Q^{*}} = I - QQ^{*} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$P_{Q_1^{*}} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}.$$

Checking solvability condition

$$P_{Q_1^{*}} f = 1 \neq 0.$$

Thus, the equation is unsolvable, but there is the pseudosolution

$$X_0^{+} = Q^{+} f, \quad X_0^{+} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \dfrac{e^{-W_1^1}}{2} \\ 0 & 0 & \dfrac{e^{-W_1^1}}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \dfrac{e^{-W_1^1}}{2} \\ \dfrac{e^{-W_1^1}}{2} \end{pmatrix}.$$

And the general solution is

$$X_t = \Phi(t,0)\big(P_{Q_r} c_r + X_0^{+}\big), \quad r = 1.$$

Since $P_Q = I - Q^{+}Q$, we obtain

$$P_Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

and

$$P_{Q_1} = \frac{1}{2} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Finally,

$$X_t = \begin{pmatrix} e^{\left(W_t^1 + W_t^2\right)} & e^{\left(W_t^1 + W_t^2\right)} + 2e^{W_t^1} - 1 & 0 \\ 0 & 1 & 0 \\ 0 & e^{W_t^1} - 1 & e^{W_t^1} \end{pmatrix} \left(\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} c_1 + \begin{pmatrix} 1 \\ \dfrac{e^{-W_1^1}}{2} \\ \dfrac{e^{-W_1^1}}{2} \end{pmatrix} \right).$$

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