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ON THE UNIQUE SOLVABILITY OF A NONLINEAR NONLOCAL BOUNDARY-VALUE PROBLEM FOR SYSTEMS OF SECOND ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS*

ПРО ЄДИНІСТЬ РОЗВ'ЯЗКУ НЕЛІНІЙНОЇ НЕЛОКАЛЬНОЇ КРАЙОВОЇ ЗАДАЧІ ДЛЯ СИСТЕМ ФУНКЦІОНАЛЬНО-ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ ДРУГОГО ПОРЯДКУ

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We establish some optimal, in a sense, general conditions sufficient for the unique solvability of the boundary-value problem for a system of nonlinear second order functional differential equations. The class of equations considered covers, in particular, neutral type equations. Concrete example is presented to illustrate the general theory.

Встановлено нові, в певному сенсі, оптимальні умови, достатні для однозначної розв'язності крайової задачі для систем нелінійних функціонально-диференціальних рівнянь другого порядку. Клас рівнянь, що досліджувалися, може частково містити в собі рівняння нейтрального типу. Наведено приклад, що демонструє отримані результати.

1. Introduction and problem statement. The aim of this paper is to establish new general conditions sufficient for the unique solvability of a nonlocal boundary-value problem for systems of nonlinear second order functional differential equations. Such problems arise in many applications and various kinds of them are widely studied in the literature (see, e.g., [9, 17] and references therein).

The paper is motivated mainly by the recent works [2, 7, 10, 14 – 16, 18, 19, 21, 22]. By using an abstract approach based upon order-theoretical considerations, we prove sufficiently general statements on the solvability of such a problem which, in particular, extend several results of [16, 21] that have been obtained directly by techniques of calculus. The idea of proof of our theorems is based on the application of an abstract result ensuring the unique solvability of an equation with an operator satisfying Lipschitz-type conditions with respect to a suitable cone.

The main results with proofs are introduced in Sections 3 and 5 correspondingly. Some results for equations without derivatives in the right-hand side are in Section 6. An example is presented in Section 7.

Here, we consider the nonlocal boundary-value problem

$$u_k''(t) = (f_k u)(t), \quad t \in [a,b], \quad k = 1, 2, \ldots, n, \tag{1.1}$$

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$$u_k'(a) = \varphi_{1k}(u), \quad k = 1, 2, \ldots, n, \tag{1.2}$$

$$u_k(a) = \varphi_{0k}(u), \quad k = 1, 2, \ldots, n, \tag{1.3}$$

where $f_k\colon W^2([a,b],\mathbb{R}^n) \to L_1([a,b],\mathbb{R})$, $k = 1, 2, \ldots, n$, are, generally speaking, nonlinear operators, $\varphi_{ik}\colon W^2([a,b],\mathbb{R}^n) \to \mathbb{R}$, $i = 0, 1$, $k = 1, 2, \ldots, n$, are nonlinear functionals defined on the space $W^2([a,b],\mathbb{R}^n)$ of vector functions with absolutely continuous components of u'.

It is worth mentioning that the right-hand side members of equations (1.1) may contain terms with derivatives and, thus, the statements presented in what follows are applicable, in particular, to neutral type functional differential equations (exception is Section 6).

2. Notation and definitions. Till the end of the paper, we fix a bounded interval $[a, b]$ and a natural number n.

(1) $\mathbb{R} := (-\infty, \infty)$; $\|x\| := \max_{1 \le i \le n} |x_i|$ for $x = (x_i)_{i=1}^n \in \mathbb{R}^n$.

(2) $L_1([a,b],\mathbb{R}^n)$ is the Banach space of all the Lebesgue integrable vector-valued functions $u\colon [a,b] \to \mathbb{R}^n$ with the standard norm

$$L_1([a,b],\mathbb{R}^n) \ni u \longmapsto \int_a^b \|u(\xi)\|\, d\xi.$$

(3) $W^k([a,b],\mathbb{R}^n)$, $k = 1, 2$, is the set of vector-valued functions $u = (u_i)_{i=1}^n\colon [a,b] \to \mathbb{R}^n$ with $u^{(k-1)}$ absolutely continuous on $[a,b]$ and the norm given by the formula

$$W^k([a,b],\mathbb{R}^n) \ni u \longmapsto \|u\|_k := \int_a^b \|u^{(k)}(\xi)\|\, d\xi + \sum_{m=0}^{k-1} \|u^{(m)}(a)\|. \tag{2.1}$$

(4) For $k = 1, 2$ and $m = \overline{0, 2}$, we put

$$W^k_{(m)}([a,b],\mathbb{R}^n) := \Big\{ u = (u_i)_{i=1}^n\colon [a,b] \to \mathbb{R}^n \in W^k([a,b],\mathbb{R}^n)\colon$$

$$\operatorname*{vrai\,min}_{t\in[a,b]} u_i^{(m)}(t) \ge 0 \quad \text{and} \quad u_i^{(j)}(a) \ge 0 \quad \text{for} \quad 0 \le j \le m-1, \quad i = 1, 2, \ldots, n \Big\}. \tag{2.2}$$

In what follows, the symbols $W^2([a,b],\mathbb{R}^n)$, $W^2_{(2)}([a,b],\mathbb{R}^n)$, etc. corresponding to the fixed a, b, and n will usually appear simply as W^2, $W^2_{(2)}$, etc.

A solution of (1.1) – (1.3), as usual, is understood in the sense of the following definition which is customary in the contemporary literature on the theory of functional-differential equations (see, e. g., [1]).

Definition 2.1. *By a solution of problem (1.1) – (1.3), we mean an absolutely continuous vector-valued function $u = (u_k)_{k=1}^n\colon [a,b] \to \mathbb{R}^n$ such that its components satisfy conditions (1.2) and (1.3) and equality (1.1) holds for almost all $t \in [a,b]$.*

We shall use a special class of linear operators. Let $h_i = (h_{ik})_{k=1}^n\colon W^2 \to \mathbb{R}$, $i = 1, 2$, be linear mappings.

Definition 2.2. *We say that a linear operator* $p = (p_k)_{k=1}^n : W^2 \to L_1$ *belongs to the set* $\mathcal{S}_{h_1,h_0}$ *if the boundary-value problem*

$$u_k''(t) = (p_k u)(t) + q_k(t), \quad t \in [a,b], \tag{2.3}$$

$$u_k'(a) = h_{1k}(u) + c_{1k}, \tag{2.4}$$

$$u_k(a) = h_{0k}(u) + c_{0k}, \quad k = 1, 2, \ldots, n, \tag{2.5}$$

has a unique solution $u = (u_k)_{k=1}^n$ *for any* $\{q_k \mid k = 1, 2, \ldots, n\} \subset L_1$ *and* $\{c_{ik} \mid k = 1, 2, \ldots, n\} \subset \subset \mathbb{R}$, $i = 0, 1$, *and, moreover, the solution of (2.3) – (2.5) possesses the property*

$$\min_{t\in[a,b]} u_k(t) \geq 0, \quad k = 1, 2, \ldots, n,$$

whenever the functions q_k, $k = 1, 2, \ldots, n$, *and the constants* c_{ik}, $i = 0, 1$, $k = 1, 2, \ldots, n$, *appearing in (2.3) – (2.5) are nonnegative.*

A number of conditions sufficient for the unique solvability of the linear problem (2.3) – (2.5) can be deduced, for example, from results of [2 – 4, 6, 8, 14, 20, 23].

Definition 2.3. *A linear operator* $p = (p_k)_{k=1}^n : W^2 \to L_1$ *is said to be positive if*

$$\operatorname*{vrai\,min}_{t\in[a,b]} (p_k u)(t) \geq 0, \quad k = 1, 2, \ldots, n,$$

for any $u = (u_k)_{k=1}^n$ *from* $W^2_{(0)}$.

The definition above describes a natural notion of positivity which means that a positive operator p transforms nonnegative elements of W^2 to almost everywhere nonnegative functions from L_1.

3. Sufficient conditions for the unique solvability. The theorem presented below provides a general condition ensuring the unique solvability of the nonlocal nonlinear boundary-value problem (1.1) – (1.3).

Theorem 3.1. *Suppose that there exist certain linear operators* $p = (p_k)_{k=1}^n : W^2 \to L_1$, $\tilde{p} = (\tilde{p}_k)_{k=1}^n : W^2 \to L_1$ *and linear functionals* $h_i = (h_{ik})_{k=1}^n : W^2 \to \mathbb{R}^n$ *and* $\tilde{h}_i = (\tilde{h}_{ik})_{k=1}^n : W^2 \to \to \mathbb{R}^n$, $i = 0, 1$, *such that for arbitrary functions* $u = (u_k)_{k=1}^n : [a,b] \to \mathbb{R}^n$, $v = (v_k)_{k=1}^n : [a,b] \to \to \mathbb{R}^n$ *from* W^2 *with the properties*

$$u_k(t) \geq v_k(t), \quad t \in [a,b], \quad k = 1, 2, \ldots, n, \tag{3.1}$$

the estimates

$$p_k(u - v)(t) \leq (f_k u)(t) - (f_k v)(t) \leq \tilde{p}_k(u - v)(t), \quad t \in [a,b], \quad k = 1, 2, \ldots, n, \tag{3.2}$$

and

$$h_{ik}(u - v) \leq \varphi_{ik}(u) - \varphi_{ik}(v) \leq \tilde{h}_{ik}(u - v), \quad k = 1, 2, \ldots, n, \quad i = 0, 1, \tag{3.3}$$

are fulfilled. Furthermore, suppose that the following inclusions are true:

$$\tilde{p} \in \mathcal{S}_{h_1,h_0}, \quad \frac{1}{2}(p + \tilde{p}) \in \mathcal{S}_{\frac{1}{2}(h_1+\tilde{h}_1), \frac{1}{2}(h_0+\tilde{h}_0)}. \tag{3.4}$$

Then the boundary-value problem (1.1) – (1.3) has a unique solution.

The following statements are true.

Theorem 3.2. *Assume that, for arbitrary functions* $u = (u_k)_{k=1}^n : [a,b] \to \mathbb{R}^n$, $v = (v_k)_{k=1}^n : [a,b] \to \mathbb{R}^n$ *from* W^2 *with properties (3.1), the inequalities*

$$|(f_k u)(t) - (f_k v)(t) - l_{1k}(u-v)(t)| \leq l_{2k}(u-v)(t), \quad k = 1, 2, \ldots, n, \tag{3.5}$$

and (3.3) are true for some linear functionals $h_i, \tilde{h}_i : W^2 \to \mathbb{R}^n$, $i = 0, 1$, *and linear operators* $l = (l_{jk})_{k=1}^n : W^2 \to L_1$, $j = 1, 2$, *satisfying the inclusions*

$$l_1 + l_2 \in \mathcal{S}_{h_1, h_0}, \quad l_1 \in \mathcal{S}_{\frac{1}{2}(h_1 + \tilde{h}_1), \frac{1}{2}(h_0 + \tilde{h}_0)}. \tag{3.6}$$

Then the boundary-value problem (1.1) – (1.3) has a unique solution.

Let us put $l_1 + l_2 = l$ and $l_1 - l_2 = 0$ then Theorem 3.2 implies the following corollary.

Corollary 3.1. *Let there exist certain linear operator* $l = (l_k)_{k=1}^n : W^2 \to L_1$ *and linear functionals* $h_i = (h_{ik})_{k=1}^n : W^2 \to \mathbb{R}^n$ *and* $\tilde{h}_i = (\tilde{h}_{ik})_{k=1}^n : W^2 \to \mathbb{R}^n$, $i = 0, 1$, *such that the inclusions*

$$l \in S_{h_1, h_0}, \quad \frac{1}{2}\, l \in \mathcal{S}_{\frac{1}{2}(h_1 + \tilde{h}_1), \frac{1}{2}(h_0 + \tilde{h}_0)} \tag{3.7}$$

hold, moreover the estimates (3.3) and

$$0 \leq (f_k u)(t) - (f_k v)(t) \leq l_k(u-v)(t), \quad t \in [a,b], \quad k = 1, 2, \ldots, n, \tag{3.8}$$

are fulfilled for any absolutely continuous functions u *and* v *from* W^2 *with property (3.1).*

Then the boundary-value problem (1.1) – (1.3) has a unique solution.

If $l_1 = 0$ and $l_2 = l$ then Theorem 3.2 takes the next form.

Corollary 3.2. *Assume that there exist certain linear functionals* $h_i = (h_{ik})_{k=1}^n : W^2 \to \mathbb{R}^n$ *and* $\tilde{h}_i = (\tilde{h}_{ik})_{k=1}^n : W^2 \to \mathbb{R}^n$, $i = 0, 1$, *with property (3.3) such that for arbitrary functions* $u = (u_k)_{k=1}^n : [a,b] \to \mathbb{R}^n$ *and* $v = (v_k)_{k=1}^n : [a,b] \to \mathbb{R}^n$ *from* W^2 *with the properties (3.1) the inequalities*

$$|(f_k u)(t) - (f_k v)(t)| \leq l_k(u-v)(t), \quad k = 1, 2, \ldots, n, \tag{3.9}$$

are true for some linear operator $l = (l_k)_{k=1}^n : W^2 \to L_1$ *satisfying the inclusion*

$$l \in S_{h_1, h_0}. \tag{3.10}$$

Then the boundary-value problem (1.1) – (1.3) has a unique solution.

Theorem 3.3. *Assume that, for any* $\{u, v\} \subset W^2$ *with property (3.1), the functionals* φ_0 *and* φ_1 *satisfy estimates (3.3) with certain linear functionals* $h_i, \tilde{h}_i : W^2 \to \mathbb{R}^n$, $i = 0, 1$.

In addition, let there exist some positive linear operators $g_i = (g_{ik})_{k=1}^n : W^2 \to L_1$, $i = 1, 2$, *and a constant* $\gamma \in (0, 1)$ *such that*

$$g_1 + (1 - 2\gamma)\, g_2 \in \mathcal{S}_{h_1, h_0}, \quad -\gamma g_2 \in \mathcal{S}_{\frac{1}{2}(h_1 + \tilde{h}_1), \frac{1}{2}(h_0 + \tilde{h}_0)}, \tag{3.11}$$

and the inequalities

$$|(f_k u)(t) - (f_k v)(t) + g_{2k}(u - v)(t)| \leq g_{1k}(u - v)(t), \quad k = 1, 2, \ldots, n, \tag{3.12}$$

hold on $[a, b]$ *for any vector-valued functions* $u = (u_k)_{k=1}^n$ *and* $v = (v_k)_{k=1}^n$ *from* W^2 *with properties (3.1).*

Then the boundary-value problem (1.1) – (1.3) has a unique solution.

Taking $\gamma = \dfrac{1}{2}$ $\left(\text{or } \gamma = \dfrac{1}{4}\right)$ Theorem 3.3 allows to obtain the next corollary.

Corollary 3.3. *Let there exist and positive linear operators* $g_i = (g_{ik})_{k=1}^n \colon W^2 \to L_1$, $i = 1, 2$, *condition (3.12) be satisfied for all* $\{u, v\} \subset W^2$ *with property (3.1). Let, moreover, (3.3) hold with certain linear functionals* $h_i, \tilde{h}_i \colon W^2 \to \mathbb{R}^n$, $i = 0, 1$, *and either*

$$g_1 \in \mathcal{S}_{h_1, h_0}, \quad -\frac{1}{2}\, g_2 \in \mathcal{S}_{\frac{1}{2}\,(h_1 + \tilde{h}_1), \frac{1}{2}\,(h_0 + \tilde{h}_0)} \tag{3.13}$$

or

$$g_1 + \frac{1}{2}\, g_2 \in \mathcal{S}_{h_1, h_0}, \quad -\frac{1}{4}\, g_2 \in \mathcal{S}_{\frac{1}{2}\,(h_1 + \tilde{h}_1), \frac{1}{2}\,(h_0 + \tilde{h}_0)} \tag{3.14}$$

are true.

Then problem (1.1) – (1.3) has a unique solution.

3.1. Optimality of conditions. Note that assumption (3.4), (3.6), (3.7), (3.10), (3.11), (3.13), (3.14) we can not replaced by their weakly versions. For example, in Theorem 3.2 inclusion (3.6) can not be replaced by the condition

$$(1 - \varepsilon)\,(l_1 + l_2) \in \mathcal{S}_{h_1, h_0}, \quad l_1 \in \mathcal{S}_{\frac{1}{2}\,(h_1 + \tilde{h}_1), \frac{1}{2}(h_0 + \tilde{h}_0)}$$

nor by the condition

$$l_1 + l_2 \in \mathcal{S}_{h_1, h_0}, \quad (1 - \varepsilon) l_1 \in \mathcal{S}_{\frac{1}{2}\,(h_1 + \tilde{h}_1), \frac{1}{2}\,(h_0 + \tilde{h}_0)},$$

where ε is an arbitrarily small positive number. In order to verified this, it is sufficient to use [20].

4. Auxiliary statements. We need the following statement on the unique solvability of an equation with Lipschitz type nonlinear terms (see [12, 13]). Let us consider the abstract operator equation

$$Fx = z, \tag{4.1}$$

where $F \colon E_1 \to E_2$ is a mapping between a normed space $\langle E_1, \|\cdot\|_{E_1} \rangle$ and a Banach space $\langle E_2, \|\cdot\|_{E_2} \rangle$ over the field $\mathbb{R}$ and z is an arbitrary element from E_2.

Let $K_i \subset E_i$, $i = 1, 2$, be cones [11]. The cones K_i, $i = 1, 2$, induce natural partial orderings of the respective spaces. Thus, for each $i = 1, 2$, we write $x \leqq_{K_i} y$ and $y \geqq_{K_i} x$ if and only if $\{x, y\} \subset E_i$ and $y - x \in K_i$.

Theorem 4.1 ([13], Theorem 49.4). *Let the cone K_2 be normal and reproducing. Furthermore, let $B_k\colon E_1 \to E_2$, $k = 1, 2$, be additive and homogeneous operators such that B_1^{-1} and $(B_1 + B_2)^{-1}$ exist and possess the properties*

$$B_1^{-1}(K_2) \subset K_1, \tag{4.2}$$

$$(B_1 + B_2)^{-1}(K_2) \subset K_1 \tag{4.3}$$

and, furthermore, let the order relation

$$\{Fx - Fy - B_1(x - y), B_2(x - y) - Fx + Fy\} \subset K_2 \tag{4.4}$$

be satisfied for any pair $(x, y) \in E_1^2$ such that $x \geqq_{K_1} y$.

Then equation (4.1) has a unique solution for an arbitrary z from E_2.

Let us recall two definitions (see, e.g., [11, 13]).

Definition 4.1. *A cone $K_2 \subset E_2$ is called normal if there exists a constant $\gamma \in (0, +\infty)$ such that $\|x\|_{E_2} \leq \gamma \|y\|_{E_2}$ for arbitrary $\{x, y\} \subset E_2$ with the property $0 \leqq_{K_2} x \leqq_{K_2} y$.*

Definition 4.2. *A cone K_1 is called generating in E_1 if every element $u \in E_1$ can be represented in the form $u = u_1 - u_2$, where $\{u_1, u_2\} \subset K_1$.*

Let us now formulate several lemmas.

Lemma 4.1. *The following propositions are true:*

(1) The set $W^2_{(0)}$ is a cone in the space W^2.

(2) The set $W^2_{(2)}$ is a normal and generating cone in the space W^2.

Proof. The assertions of Lemma 4.1 follow immediately from the definitions of the sets $W^2_{(0)}$ and $W^2_{(2)}$ (see the notation in Section 2).

For any $p\colon W^2 \to L_1$ and $h_i\colon W^2 \to \mathbb{R}^n$, $i = 0, 1$, let us define an operator $V_{p,h_1,h_0}\colon W^2 \to W^2$ by putting

$$\left(V_{p,h_1,h_0}u\right)(t) := u(t) - \int\limits_a^t \left(\int\limits_a^s (pu)(\xi) d\xi \right) ds - (t - a)h_1(u) - h_0(u) \tag{4.5}$$

for all $u \in W^2$ and $t \in [a, b]$.

Lemma 4.2. *A function u from W^2 is a solution of the equation*

$$(V_{p,h_1,h_0}u)(t) = \int\limits_a^t \left(\int\limits_a^s q(\xi) d\xi \right) ds + c_1\,(t - a) + c_0, \quad t \in [a, b],$$

where $q \in L_1$ and $c_i \in \mathbb{R}$, $i = 0, 1$, if and only if it is a solution of the nonlocal boundary-value problem (2.3) – (2.5).

The next lemma establishes the relations between the property described by Definition 2.2 and the positive invertibility of operator (4.5).

Lemma 4.3. *If a linear operator $p = (p_k)_{k=1}^n\colon W^2 \to L_1$ satisfies the inclusion*

$$p \in \mathcal{S}_{h_1,h_0}, \tag{4.6}$$

then the operator $V_{p,h_1,h_0}\colon W^2 \to W^2$ *given by formula (4.5) is invertible and, moreover, its inverse* V_{p,h_1,h_0}^{-1} *has the property*

$$V_{p,h_1,h_0}^{-1}(W^2_{(2)}) \subset W^2_{(0)}. \tag{4.7}$$

Proof. Suppose that mapping p belongs to the set $\mathcal{S}_{h_1,h_0}$. Given an arbitrary function $y = (y_k)_{k=1}^n \in W^2$, consider the equation

$$V_{p,h_1,h_0}u = y. \tag{4.8}$$

Since $y \in W^2$, we have that, in particular, y and y' are absolutely continuous. In view of (4.6), there exists a unique function $u \in W^2$ such that

$$u_k''(t) = (p_k u)(t) + y_k''(t), \quad t \in [a,b], \quad k = 1,2,\ldots,n,$$

and

$$u_k'(a) = h_{1k}(u) + y_k'(a), \quad k = 1,2,\ldots,n,$$

$$u_k(a) = h_{0k}(u) + y_k(a), \quad k = 1,2,\ldots,n.$$

By Lemma 4.2, it follows that u is a unique solution of equation (4.8). Due to the arbitrariness of $y \in W^2$, it follows that V_{p,h_1,h_0}^{-1} exists and, hence, $u = V_{p,h_1,h_0}^{-1}y$.

Moreover, inclusion (4.6) also guarantees that if the functions y_k, $k = 1,2,\ldots,n$, are such that

$$y_k''(t) \geq 0, \quad y_k'(a) \geq 0, \quad y_k(a) \geq 0, \quad t \in [a,b], \quad k = 1,2,\ldots,n, \tag{4.9}$$

then the components of u are nonnegative and, therefore, $V_{p,h_1,h_0}^{-1}y \in W^2_{(0)}$. However, relations (4.9) mean that $y \in W^2_{(2)}$. Since y is arbitrary, we thus arrive at the required inclusion (4.7).

Lemma 4.4. *The identity*

$$V_{p,h_1,h_0} + V_{\tilde{p},\tilde{h}_1,\tilde{h}_0} = 2V_{\frac{1}{2}(p+\tilde{p}),\frac{1}{2}(h_1+\tilde{h}_1),\frac{1}{2}(h_0+\tilde{h}_0)} \tag{4.10}$$

holds for arbitrary linear operators p, $\tilde{p}\colon W^2 \to L_1$ *and linear functionals* h_i, $\tilde{h}_i\colon W^2 \to \mathbb{R}^n$, $i = 0,1$.

Proof. This statement is an easy consequence of (4.5). Indeed, for any $u \in W^2$ and $t \in [a,b]$, formula (4.5) implies the equality

$$(V_{p,h_1,h_0}u)(t) + (V_{\tilde{p},\tilde{h}_1,\tilde{h}_0}u)(t) = 2\Bigg(u - \frac{1}{2}\int_a^t \Bigg(\int_a^s ((pu)(\xi) + (\tilde{p}u)(\xi))\,d\xi\Bigg)ds -$$

$$- \frac{t-a}{2}\,(h_1(u) + \tilde{h}_1(u)) - \frac{1}{2}\,(h_0(u) + \tilde{h}_0(u))\Bigg),$$

which, in view of the linearity of the operators p, $\tilde{p}$ and functionals h_i, $\tilde{h}_i$, $i = 0, 1$, leads us immediately to (4.10).

5. Proofs. ***Proof of Theorem 3.1.*** By analogy to Lemma 4.2, it is easy to see that an absolutely continuous vector-valued function $u = (u_k)_{k=1}^n \colon [a,b] \to \mathbb{R}^n$ is a solution of (1.1)–(1.3) if, and only if it satisfies the equation

$$u(t) = \int_a^t \left(\int_a^s (fu)(\xi) d\xi \right) ds + (t-a)\varphi_1(u) + \varphi_0(u), \quad t \in [a,b]. \tag{5.1}$$

Let us take $E_1 = E_2 = W^2$ and define a mapping $F \colon W^2 \to W^2$ by setting

$$F := V_{f,\varphi_1,\varphi_0}, \tag{5.2}$$

where $V_{f,\varphi_1,\varphi_0}$ is given by (4.5). Then (5.1) takes the form (4.1) with $z = 0$. We shall show that, under the conditions assumed, equation (5.1) has a unique solution.

Using notation (4.5), define the linear mappings $B_i \colon W^2 \to W^2$, $i = 1, 2$, by putting

$$B_1 := V_{\tilde{p},\tilde{h}_1,\tilde{h}_0}, \quad B_2 := V_{p,h_1,h_0}. \tag{5.3}$$

Let us also put

$$w_{u,v}(t) := (V_{f,\varphi_1,\varphi_0}u)(t) - (V_{f,\varphi_1,\varphi_0}v)(t), \quad t \in [a,b], \tag{5.4}$$

for all u and v from W^2 with properties (3.1). Then, due to (4.5),

$$w_{u,v}(a) = u(a) - v(a) - \varphi_0(u) + \varphi_0(v),$$

$$w'_{u,v}(a) = u'(a) - v'(a) - \varphi_1(u) + \varphi_1(v)$$

and, therefore, we have the componentwise inequalities

$$(B_2(u-v))(a) \le w_{u,v}(a) \le (B_1(u-v))(a), \tag{5.5}$$

$$(B'_2(u-v))(a) \le w'_{u,v}(a) \le (B'_1(u-v))(a). \tag{5.6}$$

According to (3.2), we have

$$-\tilde{p}_k(u-v)(t) \le -(f_k u)(t) + (f_k v)(t) \le -p_k(u-v)(t)$$

and, therefore, due to (4.5), the componentwise estimates

$$w_{u,v}(t) \le u(t) - v(t) - \int_a^t \left(\int_a^s p(u-v)(\xi) d\xi \right) ds - (t-a)h_1(u-v) - h_0(u-v) \tag{5.7}$$

and

$$w_{u,v}(t) \geq u(t) - v(t) - \int\limits_a^t \left(\int\limits_a^s \tilde{p}(u-v)(\xi)d\xi \right) ds - (t-a)\tilde{h}_1(u-v) - \tilde{h}_0(u-v) \tag{5.8}$$

hold for any u, v with properties (3.1) and $t \in [a, b]$.

Let us put

$$K_1 := W^2_{(0)}, \quad K_2 := W^2_{(2)}. \tag{5.9}$$

By Lemma 4.1, both sets (5.9) are cones and, moreover, K_2 is normal and generating in W^2.

According to the definition (2.2) of the set $W^2_{(2)}$, estimates (5.5) – (5.8) mean that

$$\{B_2(u-v) - w_{u,v}, w_{u,v} - B_1(u-v)\} \subset W^2_{(2)}$$

or, equivalently,

$$\{V_{f,\varphi_1,\varphi_0}u - V_{f,\varphi_1,\varphi_0}v - B_1(u-v), B_2(u-v) - V_{f,\varphi_1,\varphi_0}u + V_{f,\varphi_1,\varphi_0}v\} \subset W^2_{(2)} \tag{5.10}$$

for arbitrary u and v from W^2 with property (3.1). Thus, relation (4.4) holds with F, B_1, and B_2 given by (5.2), (5.3) and the cones K_1 and K_2 defined by (5.9).

Recalling (5.3) and applying Lemma 4.4, we obtain the identity

$$B_1 + B_2 = 2V_{\frac{1}{2}(p+\tilde{p}), \frac{1}{2}(h_1+\tilde{h}_1), \frac{1}{2}(h_0+\tilde{h}_0)}. \tag{5.11}$$

In view of assumption (3.4), Lemma 4.3 guarantees the invertibility of the operators $V_{\tilde{p},\tilde{h}_1,\tilde{h}_0}$ and $V_{\frac{1}{2}(p+\tilde{p}), \frac{1}{2}(h_1+\tilde{h}_1), \frac{1}{2}(h_0+\tilde{h}_0)}$. Consequently, we have $B_1^{-1} = V^{-1}_{\tilde{p},\tilde{h}_1,\tilde{h}_0}$ and, by (5.11), the equality

$$(B_1 + B_2)^{-1} = \frac{1}{2} V^{-1}_{\frac{1}{2}(p+\tilde{p}), \frac{1}{2}(h_1+\tilde{h}_1), \frac{1}{2}(h_0+\tilde{h}_0)}$$

holds. The same Lemma 4.3 ensures the positivity of the inverse operators in the sense that

$$V^{-1}_{\tilde{p},\tilde{h}_1,\tilde{h}_0}(W^2_{(2)}) \subset W^2_{(0)},$$

$$V^{-1}_{\frac{1}{2}(p+\tilde{p}), \frac{1}{2}(h_1+\tilde{h}_1), \frac{1}{2}(h_0+\tilde{h}_0)}(W^2_{(2)}) \subset W^2_{(0)}.$$

Therefore, inclusions (4.2) and (4.3) are true for operators (5.3) with respect to cones (5.9).

Applying Theorem 4.1, we establish the unique solvability of equation (5.1) and, hence, of the boundary-value problem (1.1) – (1.3).

Theorem 3.1 is proved.

Proof of Theorem 3.2. This statement is proved similarly to [5] (Theorem 2). It is obvious, that for arbitrary functions u and v from W^2 with property (3.1), condition (3.5) is equivalent to the relation

$$-l_{2k}(u-v)(t) + l_{1k}(u-v)(t) \leq (f_k u)(t) - (f_k v)(t) \leq l_{2k}(u-v)(t) + l_{1k}(u-v)(t) \tag{5.12}$$

for $t \in [a,b]$ and $k = 1, 2, \ldots, n$. Let us put

$$p := l_1 - l_2, \quad \tilde{p} := l_1 + l_2. \tag{5.13}$$

Then (5.12) means that f satisfies condition (3.2). It is also clear that (3.6) ensures the validity of condition (3.4) with p and $\tilde{p}$ given by (5.13). Application of Theorem 3.1 thus leads us to the assertion of Theorem 3.2.

Proof of Theorem 3.3. Taking into account conditions (3.11), (3.12), one can check that the operators $l_i \colon W^2 \to L_1$, $i = 1, 2$, defined by the formulae

$$l_1 := -\gamma g_2, \quad l_2 := g_1 + (1-\gamma) g_2 \tag{5.14}$$

satisfy conditions (3.5), (3.6) of Theorem 3.2. Indeed, estimate (3.12), the assumption that $0 < \gamma < 1$, and the positivity of the operator g_2 imply that, for any absolutely continuous functions $u = (u_k)_{k=1}^n \colon [a,b] \to \mathbb{R}^n$ and $v = (v_k)_{k=1}^n \colon [a,b] \to \mathbb{R}^n$ with properties (3.1), the relations

$$\begin{aligned}
&|(f_k u)(t) - (f_k v)(t) + \gamma g_2 (u-v)(t)| = \\
&\quad = |(f_k u)(t) - (f_k v)(t) + g_{2k}(u-v)(t) - (1-\gamma) g_{2k}(u-v)(t)| \leq \\
&\quad \leq g_{1k}(u-v)(t) + |(1-\gamma) g_{2k}(u-v)(t)| = \\
&\quad = g_{1k}(u-v)(t) + (1-\gamma)(g_{2k}(u-v)(t)), \quad t \in [a,b], \quad k = 1, 2, \ldots, n,
\end{aligned}$$

are true. This means that f satisfies estimate (3.5) with the operators l_i, $i = 1, 2$, defined by formulae (5.14). Therefore, it only remains to note that assumption (3.11) ensures the validity of inclusions (3.6) for operators (5.14). Applying Theorem 3.2, we arrive at the required assertion.

6. The case of an equation without derivatives in the right-hand side. In the general case, l from equation (1.1) is given on W^2 only and, thus, the right-hand side term of equation (1.1) may contain u'', which corresponds to an equation of neutral type.

If the operator l in equation (1.1) is defined not only on W^2 but also on the entire space W^1, then a statement equivalent to Theorem 3.1 can be obtained with the help of results established in [5].

Given an operator $p \colon W^1 \to L_1$, we put

$$(I_p u)(t) := \int_a^t (pu)(s) ds, \quad t \in [a,b], \tag{6.1}$$

for any u from W^1, so that I_p is a map from W^1 to itself. We need the following definition [5].

Definition 6.1. *Let $h \colon W^1 \to \mathbb{R}^n$ be a continuous linear vector functional. A linear operator $p \colon W^1 \to L_1$ is said to belong to the set $\mathcal{S}_h$ if the boundary-value problem*

$$u'(t) = (pu)(t) + \alpha(t), \quad t \in [a,b], \tag{6.2}$$

$$u(a) = h(u) + c \tag{6.3}$$

has a unique solution $u = (u_k)_{k=1}^n$ *for any* $\alpha = (\alpha_k)_{k=1}^n \in L_1$, $c \in \mathbb{R}^n$ *and, moreover, the solution of (6.2), (6.3) has nonnegative components provided that the functions* α_k, $k = 1, 2, \ldots, n$, *are nonnegative almost everywhere on* $[a, b]$.

In the case where the operator f, which determines the right-hand side of equation (1.1), is well defined on the entire space W^1, results of the preceding sections admit an alternative formulation. In particular, the following statements hold.

Theorem 6.1. *Suppose that there exist certain linear operators* $p = (p_k)_{k=1}^n : W^1 \to L_1$, $\tilde{p} = (\tilde{p}_k)_{k=1}^n : W^1 \to L_1$, *satisfying the inclusions*

$$I_{\tilde{p}} + h_1 \in \mathcal{S}_{h_0}, \quad \frac{1}{2}\, I_{p+\tilde{p}} + \frac{1}{2}\,(h_1 + \tilde{h}_1) \in \mathcal{S}_{\frac{1}{2}\,(h_0+\tilde{h}_0)},$$

where $h_i = (h_{ik})_{k=1}^n : W^1 \to \mathbb{R}^n$, $\tilde{h}_i = (\tilde{h}_{ik})_{k=1}^n : W^1 \to \mathbb{R}^n$, $i = 0, 1$, *are linear vector functionals, and such that inequalities (3.2) and (3.3) hold for an arbitrary* u *and* v *from* W^1 *with property (3.1).*

Then the nonlocal boundary-value problem (1.1) – (1.3) has a unique solution.

Theorem 6.2. *Assume that there exist certain linear operator* $l : W^1 \to L_1$ *and linear functionals* $h_i = (h_{ik})_{k=1}^n : W^1 \to \mathbb{R}^n$, $\tilde{h}_i = (\tilde{h}_{ik})_{k=1}^n : W^1 \to \mathbb{R}^n$, $i = 0, 1$, *satisfying the inclusion*

$$I_p + h_1 \in S_{h_0}$$

and the estimations (3.9) and (3.3) are fulfilled for any absolutely continuous functions u *and* v *with property (3.1).*

Then the boundary-value problem (1.1) – (1.3) has a unique solution.

Theorem 6.3. *Let there exist certain positive linear operators* $g_i = (g_{ik})_{k=1}^n : W^1 \to L_1$, $i = 1, 2$, *and linear functionals* $h_i = (h_{ik})_{k=1}^n : W^1 \to \mathbb{R}^n$, $\tilde{h}_i = (\tilde{h}_{ik})_{k=1}^n : W^1 \to \mathbb{R}^n$, $i = 0, 1$, *which satisfy inequalities (3.3) and (3.12) for arbitrary* u *and* v *from* W^1 *with property (3.1), and, moreover, are such that the inclusions*

$$I_{g_1} + h_1 \in \mathcal{S}_{h_0}, \quad -\frac{1}{2}\, I_{g_2} + \frac{1}{2}\,(h_1 + \tilde{h}_1) \in \mathcal{S}_{\frac{1}{2}(h_0+\tilde{h}_0)}$$

hold.

Then the nonlocal boundary-value problem (1.1) – (1.3) has a unique solution.

To prove the Theorems 6.1, 6.2 and 6.3 we use the following lemma.

Lemma 6.1. *If* $l : W^1 \to L_1$ *is a bounded linear operator, then the inclusion*

$$I_l + \theta \in \mathcal{S}_h \tag{6.4}$$

implies that $l \in \mathcal{S}_{\theta,h}$.

Proof of Lemma 6.1. According to Definition 2.2, l belongs to $\mathcal{S}_{\theta,h}$ if and only if problem (2.3) – (2.5) has a unique solution for any $q \in L_1$, $c_i \in \mathbb{R}^n$, $i = 0, 1$, and, moreover, the solution is nonnegative for nonnegative q, c_0, c_1. By integrating (2.3), we can represent problem (2.3) – (2.5) in the equivalent form

$$u'(t) = (I_l u)(t) + \theta(u) + c_1 + \int\limits_a^t q(s)ds, \quad t \in [a, b], \tag{6.5}$$

$$u(a) = h(u) + c_0, \tag{6.6}$$

which, obviously, is a particular case of (6.2), (6.3) with $p := I_l + \theta$, and $\alpha := \int_a^{\cdot} q(s)ds + c_1$. However, by virtue of Definition 6.1, the unique solvability of problem (6.5), (6.6) and the monotone dependence of its solution on q follow from inclusion (6.4). Therefore, $l \in \mathcal{S}_{\theta,h}$.

7. Example of a functional differential equations of the second order. We consider the boundary-value problem for the nonlinear scalar differential equation with argument deviations

$$u''(t) = \alpha(t)\left(d + \lambda(t)\sin\left(u(\omega(t))\right)\right)^{\frac{1}{2m+1}}, \quad t \in [a,b], \tag{7.1}$$

$$u'(a) = 0, \tag{7.2}$$

$$u(a) = \mu u(b) + c, \tag{7.3}$$

where $d \in \mathbb{R}$, $c \in \mathbb{R}$, $m \in \mathbb{N}$, $\omega\colon [a,b] \to [a,b]$ is Lesbesgue measurable function, $\{\alpha, \lambda\} \subset L_1$, are functions such that

$$t \geq \omega(t), \quad \alpha(t) \geq 0 \tag{7.4}$$

and

$$0 \leq \lambda(t) < d. \tag{7.5}$$

The next result is true.

Theorem 7.1. *Let $|\mu| < 1$ and the functions α, λ, ω satisfy the conditions (7.4), (7.5) for almost all $t \in [a,b]$, and*

$$\int\limits_a^b \left(\int\limits_a^t \frac{\alpha(s)\lambda(s)ds}{(2m+1)\left(d-\lambda(s)\right)^{\frac{2m}{2m+1}}} \right) dt < -\ln|\mu|. \tag{7.6}$$

Then the boundary-value problem (7.1), (7.2), (7.3) has a unique solution.

To prove Theorem 7.1, we use the following propositions concerning the scalar linear functional differential equation:

$$u'(t) = (pu)(t) + q(t), \quad t \in [a,b], \tag{7.7}$$

where p is a map from $C := C([a,b],\mathbb{R})$ to L_1.

We shall say that p is positive if it maps nonnegative functions from C to almost everywhere nonnegative elements of L_1.

Proposition 7.1 ([8], Corollary 2.1a). *Suppose that $|\mu| < 1$ and the operator p in scalar linear functional differential equation (7.7) is a positive Volterra operator and*

$$|\mu| \exp\left(\int\limits_a^b (p1)(s)ds\right) < 1. \tag{7.8}$$

Then the boundary-value problem (7.7), (7.3) is uniquely solvable for an arbitrary $q \in L$, $c \in \mathbb{R}$. Moreover, nonnegativity of q implies the nonnegativity of the solution.

Proof of Theorem 7.1. To prove Theorem 7.1 we use Theorem 6.2.

It is easy to see that the problem (7.1)–(7.3) is a particular case of (1.1)–(1.3) where $n = 1$ and the operator $f_1 \colon W^1 \to L_1$ given by the formula

$$(f_1 u)(t) := \alpha(t)\left(d + \lambda(t)\sin(u(\omega(t)))\right)^{\frac{1}{2m+1}}, \quad t \in [a,b], \tag{7.9}$$

$\varphi_1 := 0$, $\varphi_0 := \mu u(b) + c$ and $h_0 = \mu u(b) + c$ for any u from W^1. Using the Lagrange theorem and taking (7.5) into account, we get that the relations

$$\left|\alpha(t)\left(d + \lambda(t)\sin(u(\omega(t)))\right)^{\frac{1}{2m+1}} - \alpha(t)\left(d + \lambda(t)\sin(v(\omega(t)))\right)^{\frac{1}{2m+1}}\right| \leq$$

$$\leq \sup_{\xi\in\mathbb{R}} \frac{\alpha(t)\lambda(t)|\cos\xi|\left(u\left(\omega(t)\right) - v\left(\omega(t)\right)\right)}{(2m+1)(d + \lambda(t)\sin\xi)^{\frac{2m}{2m+1}}} \leq \frac{\alpha(t)\lambda(t)\left(u\left(\omega(t)\right) - v\left(\omega(t)\right)\right)}{(2m+1)(d - \lambda(t))^{\frac{2m}{2m+1}}}$$

hold for almost all $t \in [a,b]$ and for arbitrary absolutely continuous functions $u \colon [a,b] \to \mathbb{R}$ and $v \colon [a,b] \to \mathbb{R}$ possessing the properties (3.1).

Let us put

$$(lu)(t) := \frac{\alpha(t)\lambda(t)u(\omega(t))}{(2m+1)(d - \lambda(t))^{\frac{2m}{2m+1}}}, \quad t \in [a,b]. \tag{7.10}$$

Taking into account (7.5) and (7.4) for $u \in W^1$, we see that (3.9) is true. Now we need to make sure that $\chi \in \mathcal{S}_{h_0}$, where

$$\chi := I_l. \tag{7.11}$$

It is clear from (7.9), (7.10) that f and l can be considered as a mapping from C to L_1, so we can use Proposition 7.1. It is easy to see, that χ is a positive operator, which, due to assumption (7.4) is of Volterra type. It follows from (6.1), (7.10) and (7.11) that

$$\int_a^b (\chi 1)(t)dt = \int_a^b \left(\int_a^t \frac{\alpha(s)\lambda(s)ds}{(2m+1)(d - \lambda(s))^{\frac{2m}{2m+1}}}\right) dt$$

and, hence, for $\mu \neq 0$, assumption (7.6) implies the relation

$$\int_a^b (\chi 1)(t)dt < -\ln|\mu|.$$

This means that inequality (7.8) is fulfilled.

Applying Proposition 7.1, we show that $\chi \in \mathcal{S}_{h_0}$. Note that if $\mu = 0$, then problem (7.3), (7.7) reduces to a Cauchy problem at the point a and as is known in this case (see, e. g., [8]) the inclusion $\chi \in \mathcal{S}_{h_0}$ is guaranteed by the Volterra property of χ.

So, we have shown that all the conditions of Theorem 6.2 are fulfilled. Applying the Theorem 6.2, we complete the proof.

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