UDC 517.9

OSCILLATION CRITERIA FOR SECOND-ORDER QUASILINEAR NEUTRAL FUNCTIONAL DYNAMIC EQUATION ON TIME SCALES

КРИТЕРІЇ ОСЦИЛЯЦІЇ КВАЗІЛІНІЙНОГО НЕЙТРАЛЬНО ФУНКЦІОНАЛЬНОГО ДИНАМІЧНОГО РІВНЯННЯ ДРУГОГО ПОРЯДКУ НА ЧАСОВІЙ ҐРАТЦІ

S. H. Saker

King Saud Univ.
Riyadh 11451, Saudi Arabia
Mansoura Univ., Mansoura 35516, Egypt
e-mail: shsaker@mans.edu.eg

Our aim in this paper is to esatblish some sufficent conditions for oscillation of the second-order quasilinear neutral functional dynamic equation

$$\left(p(t)\left([y(t)+r(t)y(\tau(t))]^{\Delta}\right)^{\gamma}\right)^{\Delta}+f(t,y(\delta(t))=0,\quad t\in[t_0,\infty)_{\mathbb{T}},$$

on a time scale $\mathbb{T}$, *where* $|f(t,u)|\geq q(t)\left|u^{\beta}\right|$, r, p *and* q *are real valued* rd*-continuous positive functions defined on* $\mathbb{T}$, γ *and* $\beta>0$ *are ratios of odd positive integers. Our results do not require that* $\gamma=\beta\geq 1$, $p^{\Delta}(t)\geq 0$, $\int_{t_0}^{\infty}\left(\frac{1}{p(t)}\right)^{\frac{1}{\gamma}}\Delta t=\infty$ *and* $\int_{t_0}^{\infty}\delta^{\beta}(s)q(s)[1-r(\delta(s))]^{\beta}\Delta s=\infty$. *Some examples are considered to illustrate the main results.*

Метою статті є встановлення деяких достатніх умов осциляційності квазілінійного нейтрально функціонального динамічного рівняння

$$\left(p(t)\left([y(t)+r(t)y(\tau(t))]^{\Delta}\right)^{\gamma}\right)^{\Delta}+f(t,y(\delta(t))=0,\quad t\in[t_0,\infty)_{\mathbb{T}},$$

на часовій ґратці $\mathbb{T}$, *де* $|f(t,u)|\geq q(t)\left|u^{\beta}\right|$, r, p *та* q *— дійснозначні* rd*-неперервні додатні функції, що визначені на* $\mathbb{T}$, γ *та* $\beta>0$ *є відношеннями непарних додатних цілих чисел. Отримані результати не вимагають виконання умов* $\gamma=\beta\geq 1$, $p^{\Delta}(t)\geq 0$, $\int_{t_0}^{\infty}\left(\frac{1}{p(t)}\right)^{\frac{1}{\gamma}}\Delta t=\infty$ *та* $\int_{t_0}^{\infty}\delta^{\beta}(s)q(s)[1-r(\delta(s))]^{\beta}\Delta s=\infty$. *Наведено деякі приклади, що ілюструють основні результати.*

1. Introduction. In this paper, we consider the quasilinear neutral functional dynamic equation

$$\left(p(t)\left([y(t)+r(t)y(\tau(t))]^{\Delta}\right)^{\gamma}\right)^{\Delta}+f(t,y(\delta(t))=0,\tag{1.1}$$

on a time scale $\mathbb{T}$. Throughout this paper, we will assume the following hypotheses:

(h_1) r, p and q are real valued rd-continuous positive functions defined on $\mathbb{T}$, $0\leq r(t)<1$,

(h_2) γ is a ratio of odd positive integers, $\tau:\mathbb{T}\to\mathbb{T}$, $\delta:\mathbb{T}\to\mathbb{T}$, $\tau(t)\leq t$ for all $t\in\mathbb{T}$ and $\lim_{t\to\infty}\delta(t)=\lim_{t\to\infty}\tau(t)=\infty$,

(h_3) $f(t,u) : \mathbb{T}\times\mathbb{R}\to\mathbb{R}$ is continuous function such that $uf(t,u) > 0$ for all $u \neq 0$ and there exists a positive rd-continuous function $q(t)$ defined on $\mathbb{T}$ such that $|f(t,u)| \geq q(t)|u^{\beta}|$, where $\beta > 0$ is a ratio of odd positive integers.

We shall also consider the two cases:

$$\int_{t_0}^{\infty}\left(\frac{1}{p(t)}\right)^{\frac{1}{\gamma}}\Delta t = \infty, \tag{1.2}$$

and

$$\int_{t_0}^{\infty}\left(\frac{1}{p(t)}\right)^{\frac{1}{\gamma}}\Delta t < \infty. \tag{1.3}$$

Since we are interested in the oscillatory and asymptotic behavior of solutions of (1.1) near infinity, we assume that $\sup\mathbb{T} = \infty$, and define the time scale interval $[t_0,\infty)_{\mathbb{T}}$ by $[t_0,\infty)_{\mathbb{T}} := [t_0,\infty)\cap\mathbb{T}$. Throughout this paper these assumptions will be supposed to hold. Let $\tau^*(t) = \min\{\tau(t),\delta(t)\}$ and let $T_0 = \min\{\tau^*(t) : t \geq 0\}$ and $\tau^*_{-1}(t) = \sup\{s \geq 0 : tau^*(s) \leq t\}$ for $t \geq T_0$. Clearly if $\tau^*(t) \leq t$, then $\tau^*_{-1}(t) \geq t$ for $t \geq T_0$, $\tau^*_{-1}(t)$ is nondecreasing and coincides with the inverse of $\tau^*(t)$ when the latter exists. Throughout the paper, we will use the following notations:

$$x(t) := y(t) + r(t)y(\tau(t)), \quad x^{[1]} := p(x^{\Delta})^{\gamma}, \quad x^{[2]} := (x^{[1]})^{\Delta}. \tag{1.4}$$

By a solution of (1.1), we mean a nontrivial real-valued function y which has the properties $x \in C^1_{rd}[\tau^*_{-1}(t_0),\infty)$, and $x^{[1]} \in C^1_{rd}[\tau^*_{-1}(t_0),\infty)$ where C_r is the space of rd-continuous functions. Our attention is restricted to those solutions of (1.1) which exist on some half line $[t_y,\infty)$ and satisfy $\sup\{|y(t)| : t > t_1\} > 0$ for any $t_1 \geq t_y$. A solution y of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is called nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

Much recent attention has been given to dynamic equations on time scales (or measure chains), and we refer the reader to the landmark paper of Hilger [6] for a comprehensive treatment of the subject. Since then several authors have expounded on various aspects of this new theory [5]. A book on the subject of time scales, by Bohner and Peterson [4], summarizes and organizes much of time scale calculus.

The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus (see Kac and Cheung [9]), i.e., when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = q^{\mathbb{N}} = t : t = q^k, k \in \mathbb{N}, q > 1$.

Dynamic equations on a time scale have an enormous potential for applications such as in population dynamics. For example, it can model insect populations that are continuous while in season, die out in say winter, while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a nonoverlapping population (see [4]). There are applications of dynamic equations on time scales to *quantum mechanics, electrical engineering, neural networks, heat transfer, and combinatorics*. A recent cover story article in New Scientist [14] discusses several possible applications. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers

$\mathbb{R}$. The set of all such rd-continuous functions is denoted by $C_{rd}(\mathbb{T})$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^{\sigma}(t)$ denotes $f(\sigma(t))$.

In recent years there has been much research activity concerning the oscillation and nonoscillation of solutions of second-order neutral dynamic equations on time scales, we refer the reader to the papers [1–3, 7, 8, 11–13, 15, 16]. We note that all the results obtained in these papers are given for neutral delay dynamic equations when (1.2) holds,

$$\gamma = \beta \geq 1, \quad p^{\Delta}(t) \geq 0, \quad \text{and} \quad \int\limits_{t_0}^{\infty} \delta^{\gamma}(s)q(s)[1 - r(\delta(s))]^{\gamma}\Delta s = \infty. \tag{1.5}$$

The question now is: If it is possible to find new oscillation criteria for (1.1) without (1.5)? Our interest is to give an affirmative answer to this question and to establish some oscillation criteria for (1.1) that do not require.

$$\gamma = \beta \geq 1, \quad p^{\Delta}(t) \geq 0, \quad \int\limits_{t_0}^{\infty} \delta^{\beta}(s)q(s)[1 - r(\delta(s))]^{\beta}\Delta s = \infty. \tag{1.6}$$

The paper is organized as follows. In Section 2, we consider the case when (1.2) holds and in Section 3, we consider the case when (1.3) holds. Our results are essentially new for (1.1) even in the case when $\gamma = \beta$ and can be applied when γ and/or $\beta < 1$. Applications to equations to which previously known criteria for oscillation are not applicable are given.

2. Oscillation criteria when (1.2) holds. In this section, we establish some sufficient conditions for oscillation of (1.1) when (1.2) holds. In the subsection 2.1, we consider the case when $\delta(t) > t$ and the case when $\delta(t) \leq t$ will be considered in the subsection 2.2. To prove the main results we need the following Lemmas which play important roles in the proofs of the main results even in the case when (1.3) holds.

Lemma 2.1. *Assume that* (h_1)–(h_3), *(1.2) hold and (1.1) has a nonoscillatory solution* y *on* $[t_0, \infty)_{\mathbb{T}}$ *and* x *is defined as in (1.4). Then there exists* $T > t_0$ *such that* $x(t)x^{[1]}(t) > 0$ *for* $t \geq T$.

Proof. Assume that $y(t)$ is a positive solution of (1.1) on $[t_0, \infty)_{\mathbb{T}}$. Pick $t_1 \in [t_0, \infty)_{\mathbb{T}}$ so that $t_1 > t_0$ and so that $y(t) > 0$, $y(\tau(t)) > 0$, and $y(\delta(t)) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. (Note that in the case when $y(t)$ is negative the proof is similar, since the transformation $y(t) = -z(t)$ transforms (1.1) into the same form.) Since y is a positive solution of (1.1) and $q(t) > 0$, we have (by (h_3)) that

$$(x^{[1]}(t))^{\Delta} \leq -q(t)y^{\beta}(\delta(t)) < 0 \quad \text{for} \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{2.1}$$

Then $x^{[1]}(t)$ is strictly decreasing on $[t_1, \infty)_{\mathbb{T}}$ and of one sign. We claim that $x^{[1]}(t) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. Assume not. Then there is a $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that (note $x^{[1]}(t)$ is strictly decreasing) $x^{[1]}(t_2) = c < 0$. Then from (2.1), we have $x^{[1]}(t) \leq c$ for $t \geq t_2$ and therefore

$$x^{\Delta}(t) \leq \frac{c^{\frac{1}{\gamma}}}{p^{\frac{1}{\gamma}}(t)} \quad \text{for} \quad t \in [t_2, \infty)_{\mathbb{T}}. \tag{2.2}$$

Integrating the last inequality form t_2 to t, we find from (1.2) that

$$x(t) = x(t_2) + \int_{t_2}^{t} x^{\Delta}(s)\Delta s \le x(t_2) + c^{\frac{1}{\gamma}} \int_{t_2}^{t} \frac{\Delta s}{p^{\frac{1}{\gamma}}(s)} \to -\infty \quad \text{as} \quad t \to \infty, \tag{2.3}$$

which implies that x is eventually negative. This contradiction completes the proof.

Lemma 2.2. *Assume that* $(h_1)-(h_3)$, *(1.2) hold and (1.1) has a nonoscillatory solution* y *on* $[t_0,\infty)_{\mathbb{T}}$ *and* x *is defined as in (1.4). Then there exists* $T \ge t_0$ *such that*

$$(p(t)\left(x^{\Delta}(t)\right)^{\gamma})^{\Delta} + P(t)x^{\beta}(\delta(t)) \le 0 \quad \textit{for} \quad t \ge T, \tag{2.4}$$

where

$$P(t) = q(t)(1 - r(\delta(t)))^{\beta}. \tag{2.5}$$

Proof. Assume that $y(t)$ is a positive solution of (1.1) on $[t_0,\infty)_{\mathbb{T}}$. Pick $t_1 \in [t_0,\infty)_{\mathbb{T}}$ so that $t_1 > t_0$ and so that $y(t) > 0$, $y(\tau(t)) > 0$, $y(\tau(\tau(t))) > 0$ and $y(\delta(t)) > 0$ on $[t_1,\infty)_{\mathbb{T}}$. (Note that in the case when $y(t)$ is negative the proof is similar, since the transformation $y(t) = -z(t)$ transforms (1.1) into the same form.) Since y is a positive solution of (1.1) and $q(t) > 0$, from Lemma 2.1, we see that (note $x^{[1]}(t) > 0$ and $p(t) > 0$)

$$x(t) > 0, \quad x^{\Delta}(t) > 0 \quad \text{and} \quad \left(x^{[1]}(t)\right)^{\Delta} < 0 \quad \text{for} \quad t \ge t_1. \tag{2.6}$$

Since $\tau(t) \le t$ and $r(t) \ge 0$, we have (1.4) and (2.6) that

$$x(t) = y(t) + r(t)y(\tau(t)) \le y(t) + r(t)x(\tau(t)) \le y(t) + r(t)x(t) \quad \text{for} \quad t \ge t_1.$$

Thus $y(t) \ge (1 - r(t))x(t)$ for $t \ge t_1$. Then for $t \ge t_2$, where $t_2 > t_1$ is chosen large enough, we have

$$y(\delta(t)) \ge (1 - r(\delta(t)))x(\delta(t)). \tag{2.7}$$

From (2.1) and the last inequality, we have inequality (2.4) and this completes the proof.

2.1. Oscillation of (1.1) when $\delta(t) > t$. In this subsection, we establish some sufficient conditions for oscillation of (1.1) when (1.2) holds and $\delta(t) > t$. We introduce the following notations:

$$Q(t) := P(t)\left(\frac{p^{1/\gamma}(t)P(t,T)}{p^{1/\gamma}(t)P(t,T) + \sigma(t) - t}\right)^{\beta} \eta^{\sigma}(t), \quad P(t,T) := \int_{T}^{t} \left(\frac{1}{p(s)}\right)^{\frac{1}{\gamma}} \Delta s,$$

and

$$\eta^{\sigma}(t) := \begin{cases} 1, & \text{if} \quad \beta = \gamma, \\ c_2\left(\int_T^{\sigma(t)} \frac{1}{p^{\frac{1}{\gamma}}(s)}\Delta s\right)^{\beta-\gamma}, & \text{if} \quad \beta < \gamma, \\ c_1, & \text{if} \quad \beta > \gamma, \end{cases} \tag{2.8}$$

where $T \geq t_0$ is chosen sufficiently large and c_1 and c_2 are any positive constants. We start with the following theorem.

Theorem 2.1. *Assume that* $(h_1)-(h_3)$ *and (1.2) hold. Let* y *be a nonoscillatory solution of (1.1) and make the Riccati substitution*

$$u(t) := \frac{x^{[1]}(t)}{x^{\gamma}(t)}, \tag{2.9}$$

where x *is defined as in (1.4). Then* $u(t) > 0$ *for* $t \geq T$ *(here* T *is as in Lemma 2.2) and*

$$u^{\Delta}(t) + Q(t) + \frac{\gamma}{p^{\frac{1}{\gamma}}(t)} \left(u^{\sigma}(t)\right)^{1+\frac{1}{\gamma}} \leq 0 \quad \text{for} \quad t \in [T, \infty)_{\mathbb{T}}. \tag{2.10}$$

Proof. Let y be as above and without loss of generality, we assume that there is a $t_1 > t_0$ such that $y(t) > 0$, $y(\tau(t)) > 0$, $y(\tau(\tau(t))) > 0$ and $y(\delta(t)) > 0$ for $t \geq t_1$. Then from Lemma 2.1 and (1.4), there exists $T > t_1$ such that

$$x(t) > 0, \quad x^{[1]}(t) > 0, \quad \text{and} \quad x^{[2]}(t) < 0 \quad \text{for} \quad t \geq T.$$

By the quotient rule [4] (Theorem 1.20) and the definition of $u(t)$, we have

$$u^{\Delta}(t) = \frac{x^{\gamma}(t)x^{[2]}(t) - \left(x^{\gamma}(t)\right)^{\Delta} x^{[1]}(t)}{x^{\gamma}(t)\left(x^{\sigma}(t)\right)^{\gamma}} = \frac{x^{[2]}(t)}{\left(x^{\delta}(t)\right)^{\gamma}} \frac{\left(x^{\delta}(t)\right)^{\beta}}{\left(x^{\sigma}(t)\right)^{\gamma}} - \frac{\left(x^{\gamma}(t)\right)^{\Delta} x^{[1]}(t)}{x^{\gamma}(t)\left(x^{\sigma}(t)\right)^{\gamma}}.$$

From Lemma 2.2, we see that

$$u^{\Delta}(t) \leq -P(t)\frac{\left(x^{\delta}(t)\right)^{\beta}}{\left(x^{\sigma}(t)\right)^{\gamma}} - \frac{\left(x^{\gamma}(t)\right)^{\Delta} x^{[1]}(t)}{x^{\gamma}(t)\left(x^{\sigma}(t)\right)^{\gamma}} \quad \text{for} \quad t \geq T. \tag{2.11}$$

By the Pötzsche chain rule ([4], Theorem 1.90), if $f^{\Delta}(t) > 0$ and $\gamma > 1$ (note $f^{\sigma} \geq f$) we obtain

$$\left(f^{\gamma}(t)\right)^{\Delta} = \gamma \int_0^1 \left[f(t) + \mu h f^{\Delta}(t)\right]^{\gamma-1} f^{\Delta}(t)dh \geq$$

$$\geq \gamma \int_0^1 (f(t))^{\gamma-1} f^{\Delta}(t)dh = \gamma(f(t))^{\gamma-1} f^{\Delta}(t). \tag{2.12}$$

Also by the Pötzsche chain rule ([4], Theorem 1.90), if $f^{\Delta}(t) > 0$ and $0 < \gamma \leq 1$, we obtain

$$\left(f^{\gamma}(t)\right)^{\Delta} = \gamma \int_0^1 \left[f(t) + h\mu(t) f^{\Delta}(t)\right]^{\gamma-1} dh \; f^{\Delta}(t) \geq$$

$$\geq \gamma \int_0^1 \left(f^{\sigma}(t)\right)^{\gamma-1} dh f^{\Delta}(t) = \gamma(f^{\sigma}(t))^{\gamma-1} f^{\Delta}(t). \tag{2.13}$$

Now from (2.12) and (2.13), using $f(t) = x(t)$ and the fact that $x(t)$ is increasing and $x^{[1]}(t)$ is decreasing, we have, for $\gamma > 1$, that

$$\frac{((x(t))^{\gamma})^{\Delta} x^{[1]}(t)}{(x(t))^{\gamma}(x^{\sigma}(t))^{\gamma}} \geq \frac{\gamma\left(x^{[1]}(t)\right)^{\sigma}\left(\left(x^{[1]}(t)\right)^{\sigma}\right)^{\frac{1}{\gamma}}}{p^{\frac{1}{\gamma}} x^{\sigma}(t)(x^{\sigma}(t))^{\gamma}} = \gamma\frac{1}{p^{\frac{1}{\gamma}}(t)}(u^{\sigma}(t))^{\frac{1}{\gamma}+1}.$$

Also for $0 < \gamma \leq 1$, we have

$$\frac{(x^{\gamma}(t))^{\Delta} x^{[1]}(t)}{x^{\gamma}(t)(x^{\sigma}(t))^{\gamma}} \geq \frac{\gamma\left(x^{[1]}(t)\right)^{\sigma}\left(\left(x^{[1]}\right)^{\sigma}(t)\right)^{\frac{1}{\gamma}}}{p^{\frac{1}{\gamma}}(t)(x^{\sigma}(t))^{\gamma} x^{\sigma}(t)} = \gamma\frac{1}{p^{\frac{1}{\gamma}}(t)}(u^{\sigma}(t))^{1+\frac{1}{\gamma}}.$$

Thus

$$\frac{(x^{\gamma}(t))^{\Delta} x^{[1]}(t)}{x^{\gamma}(t)(x^{\sigma}(t))^{\gamma}} \geq \gamma\frac{1}{p^{\frac{1}{\gamma}}}(u^{\sigma}(t))^{1+\frac{1}{\gamma}} \quad \text{for} \quad \gamma > 0.$$

Substituting in (2.11), we have

$$u^{\Delta}(t) \leq -P(t)\frac{\left(x^{\delta}(t)\right)^{\beta}}{(x^{\sigma}(t))^{\gamma}} - \gamma\frac{1}{p^{\frac{1}{\gamma}}(t)}(u^{\sigma})^{1+\frac{1}{\gamma}} \quad \text{for} \quad t \geq T. \tag{2.14}$$

Next consider the coefficient of P in (2.14). Since $x^{\sigma} = x + \mu x^{\Delta}$, we have

$$\frac{x^{\sigma}(t)}{x(t)} = 1 + \mu(t)\frac{x^{\Delta}}{x(t)} = 1 + \frac{\mu(t)}{p^{\frac{1}{\gamma}}(t)}\frac{\left(x^{[1]}(t)\right)^{\frac{1}{\gamma}}}{x(t)}. \tag{2.15}$$

Also since $x^{[1]}(t)$ is decreasing, we have

$$x(t) = x(T) + \int_{T}^{t}\left(x^{[1]}(s)\right)^{\frac{1}{\gamma}}\left(\frac{1}{p(s)}\right)^{\frac{1}{\gamma}}\Delta s > \left(x^{[1]}(t)\right)^{\frac{1}{\gamma}}\int_{T}^{t}\left(\frac{1}{p(s)}\right)^{\frac{1}{\gamma}}\Delta s.$$

It follows that

$$\frac{x(t)}{\left(x^{[1]}(t)\right)^{\frac{1}{\gamma}}} \geq \int_{T}^{t}\left(\frac{1}{p(s)}\right)^{\frac{1}{\gamma}}\Delta s = P(t,T) \quad \text{for} \quad t \geq T. \tag{2.16}$$

This and (2.15) imply that

$$\frac{x^{\sigma}(t)}{x(t)} = 1 + \mu(t)\frac{x^{\Delta}(t)}{x(t)} = 1 + \frac{\mu(t)}{p^{\frac{1}{\gamma}}(t)}\frac{\left(x^{[1]}(t)\right)^{\frac{1}{\gamma}}}{x(t)} \leq \frac{p^{\frac{1}{\gamma}}(t)P(t,T) + \mu(t)}{p^{\frac{1}{\gamma}}(t)P(t,T)} \quad \text{for} \quad t \geq T.$$

Hence,

$$\frac{x(t)}{x^{\sigma}(t)} \geq \frac{p^{\frac{1}{\gamma}}(t)P(t,T)}{p^{\frac{1}{\gamma}}(t)P(t,T) + \sigma(t) - t} \quad \text{for} \quad t \geq T.$$

Thus for $t \geq T$, we have

$$\frac{x^{\delta}(t)}{x^{\sigma}(t)} = \frac{x^{\delta}(t)}{x(t)}\frac{x(t)}{x^{\sigma}(t)} \geq \left(\frac{x^{\delta}(t)}{x(t)}\right)\frac{p^{\frac{1}{\gamma}}(t)P(t,T)}{p^{\frac{1}{\gamma}}(t)P(t,T)+\sigma(t)-t}. \tag{2.17}$$

Now, since $\delta(t) > t$ and $x(t)$ is increasing, we have

$$x^{\delta}(t) > x(t). \tag{2.18}$$

This and (2.17) guarantees that

$$\frac{\left(x^{\delta}(t)\right)^{\beta}}{\left(x^{\sigma}(t)\right)^{\gamma}} \geq \left(\frac{p^{\frac{1}{\gamma}}(t)P(t,T)}{p^{\frac{1}{\gamma}}(t)P(t,T)+\sigma(t)-t}\right)^{\beta}\left(x^{\sigma}(t)\right)^{\beta-\gamma} \quad \text{for} \quad t \geq T. \tag{2.19}$$

We consider the following cases. Case (1): $\beta < \gamma$. From Lemma 2.1, since $x^{[1]}(t)$ is positive and decreasing, we see that $x^{[1]}(t) \leq x^{[1]}(t_2) = c$ for $t \geq t_2$. This implies that

$$x(\sigma(t)) \leq x(t_2) + c^{\frac{1}{\gamma}}\left(\int_{t_2}^{\sigma(t)}\frac{1}{p^{\frac{1}{\gamma}}(s)}\Delta s\right).$$

Thus

$$x^{\beta-\gamma}(\sigma(t)) > (c_2)^{\beta}\left(\int_{t_2}^{\sigma(t)}\frac{1}{p^{\frac{1}{\gamma}}(s)}\Delta s\right)^{\beta-\gamma}, \tag{2.20}$$

where $c_2 = \left(\dfrac{1}{c}\right)^{\beta}$. Case (2): $\beta = \gamma$. In this case, we see that $\left(x^{\sigma}(t)\right)^{\beta-\gamma} = 1$. Case (3): $\beta > \gamma$. In this case, since $x^{\Delta}(t) > 0$, there exists $t_2 \geq t_1$ such that $x^{\sigma}(t) > x(t) > c > 0$. This implies that $\left(x^{\sigma}(t)\right)^{\beta-\gamma} > c_1$, where $c_1 = c^{\beta-\gamma}$. Combining these three cases and using the definition of $\eta^{\sigma}(t)$, we see that

$$\left(x^{\sigma}(t)\right)^{\beta-\gamma} \geq \eta^{\sigma}(t).$$

This and (2.19) imply

$$\frac{\left(x^{\delta}(t)\right)^{\beta}}{\left(x^{\sigma}(t)\right)^{\gamma}} \geq \left(\frac{p^{\frac{1}{\gamma}}(t)P(t,T)}{p^{\frac{1}{\gamma}}(t)P(t,T)+\sigma(t)-t}\right)^{\beta}\eta^{\sigma}(t) \quad \text{for} \quad t \geq T. \tag{2.21}$$

Put (2.21) into (2.14), we obtain the inequality (2.10) and this completes the proof.

Theorem 2.2 (Leighton – Wintner type). *Assume that* (h_1) – (h_3) *and (1.2) hold. Furthermore, assume that*

$$\int_{t_0}^{\infty} Q(s)\Delta s = \infty. \tag{2.22}$$

Then every solution of (1.1) oscillates.

Proof. Suppose to the contrary and assume that y is a nonoscillatory solution of equation (1.1). Without loss of generality we may assume that $y(t) > 0$, $y(\tau(t)) > 0$, $y(\tau(\tau(t))) > 0$ and $y(\delta(t)) > 0$ for $t \geq T$ (where T is as in Theorem 2.1). We consider only this case, because the proof when $y(t) < 0$ is similar. Let u be defined as in Theorem 2.1. Then from Theorem 2.1, we see that $u(t) > 0$ for $t \geq T$ and satisfies the inequality

$$-u^{\Delta}(t) \geq Q(t) + \frac{\gamma}{p^{\frac{1}{\gamma}}(t)} \left(u^{\sigma}(t)\right)^{1+\frac{1}{\gamma}} > Q(t) \quad \text{for} \quad t \geq T. \tag{2.23}$$

From the definition of $x^{[1]}(t)$, we see that

$$x^{\Delta}(t) = \left(\frac{x^{[1]}(t)}{p(t)}\right)^{\frac{1}{\gamma}}.$$

Integrating from T to t, we obtain

$$x(t) = x(T) + \int\limits_{T}^{t} \left(\frac{1}{p(s)} x^{[1]}(s)\right)^{\frac{1}{\gamma}} \Delta s \quad \text{for} \quad t \geq T.$$

Taking into account that $x^{[1]}(t)$ is positive and decreasing, we get

$$x(t) \geq x(T) + \left(x^{[1]}(t)\right)^{\frac{1}{\gamma}} \int\limits_{T}^{t} \left(\frac{1}{p(s)}\right)^{\frac{1}{\gamma}} \Delta s \quad \text{for} \quad t \geq T.$$

It follows that

$$u(t) = \frac{x^{[1]}(t)}{x^{\gamma}(t)} \leq \left(\int\limits_{t_0}^{t} \left(\frac{1}{p(s)}\right)^{\frac{1}{\gamma}} \Delta s\right)^{-\gamma} \quad \text{for} \quad t \in [T, \infty)_{\mathbb{T}},$$

which implies using (1.2) that $\lim_{t\to\infty} u(t) = 0$. Integrating (2.23) from T to ∞ and using $\lim_{t\to\infty} u(t) = 0$, we obtain

$$u(T) \geq \int\limits_{T}^{\infty} Q(s)\Delta s,$$

which contradicts (2.22). The proof is complete.

In the following, we consider the case when

$$\int\limits_{t_0}^{\infty} Q(s)\Delta s < \infty. \tag{2.24}$$

Theorem 2.3. *Assume that* $(h_1)-(h_3)$ *and (1.2) hold. Furthermore assume that there exists a positive* rd*-continuous* Δ*-differentiable function* $\phi(t)$ *such that*

$$\lim_{t\to\infty}\sup\int_{t_0}^{t}\left[\phi(s)Q(s)-\frac{p(s)((\phi^{\Delta}(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\phi^{\gamma}(s)}\right]\Delta s=\infty. \tag{2.25}$$

Then every solution of (1.1) oscillates.

Proof. Suppose to the contrary and assume that y is a nonoscillatory solution of equation (1.1). Without loss of generality we may assume that $y(t)>0$, $y(\tau(t))>0$, $y(\tau(\tau(t)))>0$ and $y(\delta(t))>0$ for $t\geq T$ (where T is as in Theorem 2.1). We consider only this case, because the proof when $y(t)<0$ is similar. Let u be defined as in Theorem 2.1. Then from Theorem 2.1, we see that $u(t)>0$ for $t\geq T$ and satisfies the inequality (2.10). From (2.10), we have

$$u^{\Delta}(t)\leq -Q(t)-\frac{\gamma}{p^{\frac{1}{\gamma}}(t)}\left(u^{\sigma}\right)^{\frac{\gamma+1}{\gamma}}\quad\text{for}\quad t\geq T. \tag{2.26}$$

Multiplying (2.26) by $\phi(s)$ and integrating from T to t $(t\geq T)$, we have

$$\int_{T}^{t}\phi(s)Q(s)\Delta s\leq -\int_{T}^{t}\phi(s)u^{\Delta}(s)\Delta s-\int_{T}^{t}\frac{\gamma\phi(s)}{p^{\frac{1}{\gamma}}(s)}\left(u^{\sigma}\right)^{\frac{\gamma+1}{\gamma}}\Delta s.$$

Using integration by parts, we get

$$\int_{T}^{t}\phi(s)Q(s)\Delta s\leq u(T)\phi(T)+\int_{T}^{t}\phi^{\Delta}(s)u^{\sigma}(s)\Delta s-\int_{t_1}^{t}\frac{\gamma\phi(s)}{p^{\frac{1}{\gamma}}(s)}\left(u^{\sigma}\right)^{\frac{\gamma+1}{\gamma}}\Delta s.$$

Setting $B=\phi^{\Delta}(s)$ and $A=\gamma\phi(s)p^{-1/\gamma}(s)$ and $u=u^{\sigma}$, and applying the inequality

$$Bu-Au^{\frac{\gamma+1}{\gamma}}\leq\frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{B^{\gamma+1}}{A^{\gamma}},$$

we have

$$\int_{T}^{t}\phi(s)Q(s)\Delta s\leq u(t_2)\phi(T)+\int_{t_1}^{t}\frac{p(s)(\phi^{\Delta}(s))^{\gamma+1}(s)}{(\gamma+1)^{\gamma+1}\phi^{\gamma}(s)}\Delta s,$$

i.e.,

$$\int_{t_2}^{t}\left[\phi(s)Q(s)-\frac{p(s)(\phi^{\Delta}(s))^{\gamma+1}(s)}{(\gamma+1)^{\gamma+1}\phi^{\gamma}(s)}\right]\Delta s<\phi(T)u(T),$$

which contradicts condition (2.25). Then every solution of (1.1) oscillates. The proof is complete.

From Theorem 2.3, we can obtain different conditions for the oscillation of (1.1) by using different choices of $\phi(t)$. For instance, if $\phi(t)=t$, we have the following result.

Corollary 2.1. *Assume that* (h_1) – (h_3) *and (1.2) hold. Furthermore, assume that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[sQ(s) - \frac{p(s)}{(\gamma+1)^{\gamma+1}s^{\gamma}} \right] \Delta s = \infty. \tag{2.27}$$

Then every solution of (1.1) oscillates.

Another method of choosing test functions can be developed by considering the function class $\Re$ which consists of kernels of two variables. Following Saker [11], we say that a function $H \in \Re$ provided H is defined for $t_0 \le s \le t$, $t, s \in [t_0, \infty)_{\mathbb{T}}$, $H(t,s) \ge 0$, $H(t,t) = 0$ for $t \ge s \ge t_0$, and for each fixed t, $H^{\Delta_i}(t,s)$ is delta integrable with respect to variable i, $i = 1, 2$. Important examples of H when $\mathbb{T} = \mathbb{R}$ are $H(t,s) = (t-s)^m$ for $m \ge 1$. When $\mathbb{T} = \mathbb{Z}$, $H(t,s) = (t-s)^{\underline{k}}$, $k \in \mathbb{N}$, where $t^{\underline{k}} = t(t-1)\ldots(t-k+1)$.

The following theorem gives new oscillation criteria for (1.1) which can be considered as an extension of Kamenev-type oscillation criterion. The proof is similar to that of the proof in [11] (Theorem 3.3), if one uses the inequality (2.10) and hence is omitted.

Theorem 2.4. *Assume that* (h_1) – (h_3) *and (1.2) hold. Let* $\phi(t)$ *be defined as in Theorem 2.3,* $H \in \Re$, *and for* $t > s$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\phi(s)Q(s) - \frac{p(s)((\phi^{\Delta}(s))^{\gamma+1}(H^{\Delta_s}(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\phi^{\gamma}(s)H^{\gamma}(t,s)} \right] \Delta s = \infty. \tag{2.28}$$

Then every solution of (1.1) oscillates.

With appropriate choices of the functions H one can establish a number of oscillation criteria for (1.1) on different types of time scales. For instance if there exists a function $h(t,s) \in \Re$ such that

$$H^{\Delta_s}(t,s) := -h(t,s)H^{\frac{\gamma}{1+\gamma}}(t,s), \tag{2.29}$$

we have from Theorem 2.4 the following oscillation result.

Corollary 2.2. *Assume that* (h_1) – (h_3) *and (1.2) hold. Let* $\phi(t)$ *be defined as in Theorem 2.3,* $H \in \Re$, *and for* $t > s$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\phi(s)Q(s) - \frac{p(s)((\phi^{\Delta}(s))^{\gamma+1}(h(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\phi^{\gamma}(s)} \right] \Delta s = \infty.$$

Then every solution of equation (1.1) is oscillatory.

As a special case by choosing $H(t,s) = (t-s)^m$ for $m \ge 1$, we have from Corollary 2.2 the following Kamenev-type oscillation criterion.

Corollary 2.3. *Assume that* (h_1) – (h_3) *and (1.2) hold. If for* $m > 1$

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_{t_0}^{t} \left[(t-s)^m Q(s) - \frac{m^{\gamma+1}p(s)((t-s)^{m-1})^{\gamma+1}}{(\gamma+1)^{\gamma+1}(t-s)^{m\gamma}} \right] \Delta s = \infty,$$

then every solution of (1.1) oscillates.

In the following, we give an example to illustrate the results in this subsection. To obtain conditions for oscillation we will use the facts

$$\int_{t_0}^{\infty} \frac{\Delta s}{s^{\nu}} = \infty, \quad \text{if} \quad 0 \le \nu \le 1, \quad \text{and} \quad \int_{t_0}^{\infty} \frac{\Delta s}{s^{\nu}} < \infty, \quad \text{if} \quad \nu > 1. \tag{2.30}$$

For more details we refer the reader to [4] (Theorem 5.68 and Corollary 5.71).

Example 2.1. Consider the following second-order neutral dynamic equation:

$$\left[y(t) + \frac{1}{2}y(\tau(t))\right]^{\Delta\Delta} + \frac{\lambda\,(\sigma(t)-1)}{t^3}y(\delta(t)) = 0 \quad \text{for} \quad t \in [2,\infty)_{\mathbb{T}}, \tag{2.31}$$

where $\mathbb{T}$ is a time scale such that $\int_1^{\infty}(\sigma(s)/s^3)\Delta s < \infty$. Here $\gamma = 1$, $\tau(t) < t$, and $\delta(t) > t$, $\tau(t)$ and $\delta(t) \in \mathbb{T}$ and $\lim_{t\to\infty}\delta(t) = \lim_{t\to\infty}\tau(t) = \infty$, $r(t) = 1/2$, $p(t) = 1$, $f(t,u) = q(t)u$, where

$$q(t) = \frac{\lambda\,(\sigma(t)-1)}{t^3},$$

and $\lambda > 0$ is a constant. Now take any $T \ge 2$, and since $p(t) = 1$, we have $P(t,T) = P(t,T) = t - T$. This gives

$$Q(t) := P(t)\frac{P(t,T)}{P(t,T)+\sigma(t)-t} = \frac{\lambda\,(\sigma(t)-1)}{2t^3}\,\frac{t-T}{t-T+\sigma(t)-t} = \frac{\lambda\,(\sigma(t)-1)}{t^3}\,\frac{t-T}{\sigma(t)-T}.$$

It is easy to see that assumptions (h_1)–(h_3) hold and also (2.24) is satisfied, since

$$\int_{t_0}^{\infty} Q(s)\Delta s = \frac{\lambda}{2}\int_{t_0}^{\infty}\frac{(\sigma(s)-1)}{s^3}\,\frac{s-T}{\sigma(s)-T}\Delta s \le \frac{\lambda}{2}\int_{2}^{\infty}\frac{\sigma(s)}{s^3} - \frac{1}{s^3}\Delta s < \infty.$$

To apply Corollary 2.1, it remains to discuss condition (2.27). Note

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[sQ(s) - \frac{r(s)}{(\gamma+1)^{\gamma+1}s^{\gamma}}\right]\Delta s =$$

$$= \limsup_{t\to\infty}\int_{2}^{t}\left(\frac{\lambda s\,(\sigma(s)-1)}{2s^3}\,\frac{s-T}{\sigma(s)-T} - \frac{1}{4s}\right)\Delta s >$$

$$> \limsup_{t\to\infty}\int_{t}^{t}\left(\frac{\lambda s^2}{2s^3} - \frac{T}{2s^2\,(s-1)} - \frac{1}{4s}\right)\Delta s = \infty,$$

provided that $\lambda > 1/2$. Hence, by Corollary 2.1 every solution of (2.31) oscillates if $\lambda > 1/2$.

2.2. Oscillation criteria when $\delta(t) \leq t$. In this subsection, we establish some sufficient conditions for oscillation of (1.1) when $\delta(t) \leq t$. We will use the following notation:

$$A(t) := P(t)\alpha^{\beta}(t)\eta^{\sigma}(t),$$

where $\eta^{\sigma}(t)$ is defined as in (2.8), and

$$\alpha(t) := \frac{p^{\frac{1}{\gamma}}(t)P(\delta(t),T)}{p^{\frac{1}{\gamma}}(t)P(t,T)+\mu(t)}, \quad \text{where} \quad P(u,v) := \int_{v}^{u} \frac{1}{p^{\frac{1}{\gamma}}(s)}\Delta s.$$

Theorem 2.5. *Assume that $(h_1)-(h_3)$ and (1.2) hold. Let y be a nonoscillatory solution of (1.1) and make the Riccati substitution*

$$w(t) := \frac{x^{[1]}(t)}{x^{\gamma}(t)}, \tag{2.32}$$

where x is defined as in (1.4). Then $w(t) > 0$ for $t \geq T$ (here T is as in Lemma 2.2) and

$$w^{\Delta}(t) + A(t) + \gamma\frac{1}{p^{\frac{1}{\gamma}}(t)}\left(w^{\sigma}\right)^{1+\frac{1}{\gamma}}(t) \leq 0 \quad \textit{for} \quad t \in [T,\infty)_{\mathbb{T}}. \tag{2.33}$$

Proof. Let y be as above and, without loss of generality, we assume that there is $t_1 > t_0$ such that $y(t) > 0$, $y(\tau(t)) > 0$, $y(\tau(\tau(t))) > 0$ and $y(\delta(t)) > 0$ for $t \geq t_1$. From the definition of w, by the quotient rule [4] (Theorem 1.20) and as in the proof of Theorem 2.1, we get

$$w^{\Delta}(t) \leq -P(t)\frac{\left(x^{\delta}(t)\right)^{\beta}}{\left(x^{\sigma}(t)\right)^{\gamma}} - \gamma\frac{1}{p^{\frac{1}{\gamma}}(t)}\left(w^{\sigma}(t)\right)^{1+\frac{1}{\gamma}} \quad \text{for} \quad t \geq T. \tag{2.34}$$

Now, we consider the coefficient of $P(t)$ in (2.34). Since $x^{[1]}(t) = p\left(x^{\Delta}\right)^{\gamma}(t)$ is decreasing for $t \geq T$, we have

$$x^{\sigma}(t) - x(\delta(t)) = \int_{\delta(t)}^{\sigma(t)} \frac{x^{[1]}(s)}{p^{\frac{1}{\gamma}}(s)}\Delta s \leq x^{[1]}(\delta(t))\int_{\delta(t)}^{\sigma(t)} \frac{1}{p^{\frac{1}{\gamma}}(s)}\Delta s,$$

and this implies that

$$\frac{x^{\sigma}(t)}{x(\delta(t))} \leq 1 + \frac{x^{[1]}(\delta(t))}{x(\delta(t))}\int_{\delta(t)}^{\sigma(t)} \frac{1}{p^{\frac{1}{\gamma}}(s)}\Delta s. \tag{2.35}$$

On the other hand, we have that

$$x(\delta(t)) > x(\delta(t)) - x(T) = \int_{T}^{\delta(t)} \frac{x^{[1]}(s)}{p^{\frac{1}{\gamma}}(s)}\Delta s \geq (x^{[1]})(\delta(t))\int_{T}^{\delta(t)} \frac{1}{p^{\frac{1}{\gamma}}(s)}\Delta s,$$

which leads to

$$\frac{x^{[1]}(\delta(t))}{x(\delta(t))} < \left(\int\limits_{T}^{\delta(t)} \frac{1}{p^{\frac{1}{\gamma}}(s)}\Delta s\right)^{-1}.$$

From this and (2.35), we get that

$$\frac{x^{\sigma}(t)}{x(\delta(t))} < 1 + \frac{\int_{\delta(t)}^{\sigma(t)} p^{-\frac{1}{\gamma}}(s)\Delta s}{\int_{T}^{\delta(t)} p^{-\frac{1}{\gamma}}(s)\Delta s} = \frac{\int_{T}^{\sigma(t)} p^{-\frac{1}{\gamma}}(s)\Delta s}{\int_{T}^{\delta(t)} p^{-\frac{1}{\gamma}}(s)\Delta s} =$$

$$= \frac{\int_{T}^{t} p^{-\frac{1}{\gamma}}(s)\Delta s + \int_{t}^{\sigma(t)} p^{-\frac{1}{\gamma}}(s)\Delta s}{\int_{T}^{\delta(t)} p^{-\frac{1}{\gamma}}(s)\Delta s} =$$

$$= \frac{\int_{T}^{t} p^{-\frac{1}{\gamma}}(s)\Delta s + \mu(t)p^{-\frac{1}{\gamma}}(t)}{\int_{T}^{\delta(t)} p^{-\frac{1}{\gamma}}(s)\Delta s} = \frac{1}{\alpha(t)} \quad \text{for} \quad t \geq T,$$

where we used the fact that, $\int_{t}^{\sigma(t)} f(s)\Delta s = \mu(t)f(t)$. Hence, we get

$$x(\delta(t)) \geq \alpha(t)x^{\sigma}(t) \quad \text{for} \quad t \geq T. \tag{2.36}$$

This implies that

$$\frac{\left(x^{\delta}(t)\right)^{\beta}}{\left(x^{\sigma}(t)\right)^{\gamma}} \geq (\alpha(t))^{\beta}\left(x^{\sigma}(t)\right)^{\beta-\gamma} \quad \text{for} \quad t \geq T.$$

As in the proof of Theorem 2.1, since $(x^{\sigma}(t))^{\beta-\gamma} \geq \eta^{\sigma}(t)$, we have

$$\frac{\left(x^{\delta}(t)\right)^{\beta}}{\left(x^{\sigma}(t)\right)^{\gamma}} \geq (\alpha(t))^{\beta}\eta^{\sigma}(t) \quad \text{for} \quad t \geq T. \tag{2.37}$$

Substituting (2.37) into (2.34), we have the desired inequality (2.33). This completes the proof.

Theorem 2.6 (Leighton – Wintner type). *Assume that (h_1) – (h_3) and (1.2) hold. Furthermore, assume that*

$$\int\limits_{t_0}^{\infty} A(s)\Delta s = \infty. \tag{2.38}$$

Then every solution of (1.1) oscillates.

Proof. Suppose the contrary and assume that y is a nonoscillatory solution of equation (1.1). Without loss of generality we may assume that $y(t) > 0$, $y(\tau(t)) > 0$, $y(\tau(\tau(t))) > 0$ and $y(\delta(t)) > 0$ for $t \geq T$ (where T is as in Theorem 2.5). We consider only this case, because the proof when $y(t) < 0$ is similar. Let w be defined as in Theorem 2.2. Then from Theorem 2.5, we see that $w(t) > 0$ for $t \geq T$ and satisfies the inequality (2.33). From (2.33), we have

$$-w^{\Delta}(t) \geq A(t) + \frac{\gamma}{p^{\frac{1}{\gamma}}(t)}\left(w^{\sigma}(t)\right)^{1+\frac{1}{\gamma}} > Q(t) \quad \text{for} \quad t \geq T. \tag{2.39}$$

From the definition of $x^{[1]}(t)$, we see that

$$x^{\Delta}(t) = \left(\frac{x^{[1]}(t)}{p(t)}\right)^{\frac{1}{\gamma}}.$$

Integrating from T to t, we obtain

$$x(t) = x(T) + \int\limits_T^t \left(\frac{1}{p(s)} x^{[1]}(s)\right)^{\frac{1}{\gamma}} \Delta s \quad \text{for} \quad t \geq T.$$

Taking into account that $x^{[1]}(t)$ is positive and decreasing, we get

$$x(t) \geq x(T) + \left(x^{[1]}(t)\right)^{\frac{1}{\gamma}} \int\limits_T^t \left(\frac{1}{p(s)}\right)^{\frac{1}{\gamma}} \Delta s \quad \text{for} \quad t \geq T.$$

It follows that

$$w(t) = \frac{x^{[1]}(t)}{x^{\gamma}(t)} \leq \left(\int\limits_{t_0}^t \left(\frac{1}{p(s)}\right)^{\frac{1}{\gamma}} \Delta s\right)^{-\gamma} \quad \text{for} \quad t \in [T, \infty)_{\mathbb{T}},$$

which implies using (1.2) that $\lim_{t\to\infty} w(t) = 0$. Integrating (2.39) from T to ∞ and using $\lim_{t\to\infty} w(t) = 0$, we obtain

$$w(T) \geq \int\limits_T^{\infty} A(s) \Delta s,$$

which contradicts (2.38). The proof is complete.

In the following we consider the case when

$$\int\limits_{t_0}^{\infty} A(s) \Delta s < \infty, \tag{2.40}$$

and proceed as in the proof of Theorem 2.3 (use the inequality (2.33)) to get the following results.

Theorem 2.7. *Assume that* $(h_1)-(h_3)$ *and (1.2) hold. Furthermore assume that there exists a positive* rd*-continuous* Δ*-differentiable function* $\phi(t)$ *such that*

$$\limsup_{t\to\infty} \int\limits_{t_0}^t \left[\phi(s)A(s) - \frac{p(s)((\phi^{\Delta}(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\phi^{\gamma}(s)}\right] \Delta s = \infty. \tag{2.41}$$

Then every solution of (1.1) oscillates.

Theorem 2.8. *Assume that (h_1)–(h_3) and (1.2) hold. Let $\phi(t)$ be defined as in Theorem 2.3, $H \in \Re$, and for $t > s$*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\phi(s)A(s) - \frac{p(s)((\phi^{\Delta}(s))^{\gamma+1}(H^{\Delta_s}(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\phi^{\gamma}(s)H^{\gamma}(t,s)} \right] \Delta s = \infty. \tag{2.42}$$

Then every solution of (1.1) oscillates.

With appropriate choices of the functions H one can establish a number of oscillation criteria for (1.1) on different types of time scales. For instance if there exists a function $h(t,s) \in \Re$ such that (2.29) holds, we have from Theorem 2.8 the following oscillation result.

Corollary 2.4. *Assume that (h_1)–(h_3) and (1.2) hold. Let $\phi(t)$ be defined as in Theorem 2.3, $H \in \Re$, and for $t > s$*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\phi(s)A(s) - \frac{p(s)((\phi^{\Delta}(s))^{\gamma+1}(h(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\phi^{\gamma}(s)} \right] \Delta s = \infty. \tag{2.43}$$

Then every solution of equation (1.1) oscillates.

As a special case by choosing $H(t,s) = (t-s)^m$ for $m \geq 1$, we have from Corollary 2.2 the following Kamenev-type oscillation criterion.

Corollary 2.5. *Assume that (h_1)–(h_3) and (1.2) hold. If for $m > 1$*

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_{t_0}^{t} \left[(t-s)^m A(s) - \frac{m^{\gamma+1}p(s)((t-s)^{m-1})^{\gamma+1}}{(\gamma+1)^{\gamma+1}(t-s)^{m\gamma}} \right] \Delta s = \infty, \tag{2.44}$$

then every solution of (1.1) oscillates.

In the following, we give an example to illustrate the results. To obtain the conditions for oscillation we will use the facts in (2.30).

Example 2.2. Assume that $\mathbb{T} = \mathbb{R}$ and consider the second-order neutral dynamic equation

$$\left(\frac{1}{t^2} \left(\left(y(t) + \frac{\delta^{-1}(t)-1}{\delta^{-1}(t)} y(\tau(t)) \right)' \right)^{\gamma} \right)' + \frac{\lambda}{t} y^{\gamma}(\delta(t)) = 0, \quad t \in [1,\infty)_{\mathbb{R}}, \tag{2.45}$$

where $\gamma > 0$ and is a ratio of odd positive integers, $\tau(t)$ and $\delta(t) \in \mathbb{T}$ and $\lim_{t\to\infty} \delta(t) = \lim_{t\to\infty} \tau(t) = \infty$, and $\tau(t) \leq t$, $\delta(t) \leq t$ and we assume that $\delta^{-1}(t)$ (the inverse of the function $\delta(t)$) exists. Here $\gamma = \beta > 0$,

$$p(t) = \frac{1}{t^2}, \quad r(t) = \frac{\delta^{-1}(t)-1}{\delta^{-1}(t)} = 1 - \frac{1}{\delta^{-1}(t)}, \quad \text{and} \quad q(t) = \frac{\lambda}{t}, \quad \lambda > 0.$$

This gives (noting $\alpha(t) = 1$, and $\eta^{\sigma}(t) = 1$) that

$$A(t) = P(t) = q(t)(1 - r(\delta(t))^{\gamma} = \frac{\lambda}{t^{\gamma+1}}.$$

We apply Theorem 2.7. It is easy to see that the assumptions $(h_1)-(h_3)$, and (1.2) hold, since

$$\int_{t_0}^{\infty}\left(\frac{1}{p(t)}\right)^{\frac{1}{\gamma}}\Delta t = \int_{t_0}^{\infty} t^{\frac{2}{\gamma}}dt = \infty.$$

Also (2.40) is satisfied, since

$$\int_{t_0}^{\infty} A(s)\Delta s = \lambda\int_{t_0}^{\infty}\frac{1}{s^{\gamma+1}}ds < \infty.$$

Finally we discuss (2.41). Note, by choosing $\phi(t) = t^{\gamma}$, that

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\phi(s)A(s) - \frac{p(s)((\phi^{\Delta}(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\phi^{\gamma}(s)}\right]\Delta s =$$

$$= \limsup_{t\to\infty}\int_{t_0}^{t}\left[s^{\gamma}\frac{\lambda}{s^{\gamma+1}} - \frac{\gamma^{\gamma+1}(s^{\gamma-1})^{\gamma+1}}{(\gamma+1)^{\gamma+1}\,(s^{\gamma})^{\gamma}\,s^2}\right]ds =$$

$$= \limsup_{t\to\infty}\int_{t_0}^{t}\left[\frac{\lambda}{s} - \frac{\gamma^{\gamma+1}}{(\gamma+1)^{\gamma+1}s^3}\right]ds = \infty,$$

provided that $\lambda > 0$. Then by Theorem 2.7 every solution of (2.45) oscillates if $\lambda > 0$. Note that none of the results established in [1–3, 7, 8, 11–13, 15, 16] can be applied to (2.45), since $p^{\Delta}(t) = p'(t) = -\dfrac{2}{t^3} < 0$.

3. Oscillation criteria when (1.3) holds. In this section, we consider the case when $\delta(t) \le \tau(t) \le t$ and (1.3) holds and establish some sufficient conditions for oscillation of (1.1). We will use the following notations:

$$g(t) := q(t)(1-r(t))^{\beta}, \quad \pi(t) := \int_{t}^{\infty}\left(\frac{1}{p(s)}\right)^{\frac{1}{\gamma}}\Delta s.$$

Remark 3.1. We note from the proof of Lemma 2.1 that if (1.2) holds, then the case $x(t)x^{[1]}(t) < 0$ is disregarded and $x(t)x^{[1]}(t) > 0$ for $t \ge T$. So if (1.2) does not hold, i.e., when (1.3) holds, we see that if y is a nonoscillatory solution of (1.1) on $[t_0,\infty)_{\mathbb{T}}$ and x is defined as in (1.4), then $x^{[1]}(t)$ is of one sign and there exists $T > t_0$ (where $T \ge t_0$ is chosen sufficiently large) such that

$$x(t)x^{[1]}(t) > 0 \quad \text{for} \quad t \ge T, \tag{3.1}$$

or

$$x(t)x^{[1]}(t) < 0 \quad \text{for} \quad t \geq T. \tag{3.2}$$

To prove the main results in this section when (1.3) holds, we need the following lemma.

Lemma 3.1. *Assume that $(h_1)-(h_3)$, (1.3) hold, $\tau^{\Delta}(t) \geq 0$ and $r^{\Delta}(t) \geq 0$. Suppose that (1.1) has a nonoscillatory solution y on $[t_0, \infty)_{\mathbb{T}}$ and x is defined as in (1.4) such that (3.2) holds. Then there exists $T \geq t_0$ such that*

$$(p(t)\left(x^{\Delta}(t)\right)^{\gamma})^{\Delta} + g(t)x^{\beta}(t) \leq 0 \quad \textit{for} \quad t \geq T. \tag{3.3}$$

Proof. Assume that $y(t)$ is a positive solution of (1.1) on $[t_0, \infty)_{\mathbb{T}}$. Pick $t_1 \in [t_0, \infty)_{\mathbb{T}}$ so that $t_1 > t_0$ and so that $y(t) > 0$, $y(\tau(t)) > 0$, $y(\tau(t)) > 0$ and $y(\delta(t)) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. (Note that in the case when $y(t)$ is negative the proof is similar, since the transformation $y(t) = -z(t)$ transforms (1.1) into the same form.) Since y is a positive solution of (1.1) and $q(t) > 0$, we have

$$(x^{[1]}(t))^{\Delta} \leq -q(t)y^{\beta}(\delta(t)) < 0 \quad \text{for} \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{3.4}$$

Then $x^{[1]}(t)$ is strictly decreasing on $[t_1, \infty)_{\mathbb{T}}$ and of one sign. Since y is a positive solution of (1.1) and $q(t) > 0$, and (3.2) holds, we see that (note $x^{[1]}(t) < 0$ and $p(t) > 0$)

$$x(t) > 0, \quad x^{\Delta}(t) < 0, \quad \text{and} \quad \left(x^{[1]}(t)\right)^{\Delta} < 0 \quad \text{for} \quad t \geq t_1. \tag{3.5}$$

Since $x(t)$ is decreasing, we may assume without loss of generality that $y(t)$ is also decreasing. If this is not the case, i.e., $y(t)$ and $y(\tau)$ are increasing for $t \geq t_1$, we see that $x(t)$ is also increasing for $t \geq t_1$, since

$$x^{\Delta}(t) = y^{\Delta}(t) + r^{\Delta}(t)y(\tau(t)) + r^{\sigma}(y(\tau(t))^{\Delta} > y^{\Delta}(t) > 0$$

(note $r(t) \geq 0$ and $r^{\Delta}(t) \geq 0$), which is a contradiction with $x^{\Delta}(t) < 0$ for $t \geq t_1$. This implies from (1.4) and (2.6) (note $x(t) > y(t)$) that

$$x(t) = y(t) + r(t)y(\tau(t)) \leq y(\tau(t)) + r(t)x(\tau(t)) \leq y(\tau(t))[1 + r(t)] \quad \text{for} \quad t \geq t_1.$$

Thus

$$y(\tau(t)) \geq \frac{x(t)}{1 + r(t)} \quad \text{for} \quad t \geq t_1.$$

Since $0 \leq r(t) < 1$, we have $1 \geq 1 - r^2(t)$, which implies that $1/(1 + r(t)) \geq (1 - r(t))$. Therefore

$$y(\tau(t)) \geq x(t)(1 - r(t)) \quad \text{for} \quad t \geq t_1.$$

Since $\delta(t) \leq \tau(t)$ for $t \geq t_2$, where $t_2 > t_1$ is chosen large enough (note $y(t)$ is decreasing), we have

$$y(\delta(t)) \geq (1 - r(t))x(t) \quad \text{for} \quad t \geq t_2. \tag{3.6}$$

From (3.4) and the last inequality, we have inequality (3.3) and this completes the proof.

Theorem 3.1. *Assume that* $(h_1)-(h_3)$, *(1.3) hold,* $\tau^{\Delta}(t) \geq 0$ *and* $r^{\Delta}(t) \geq 0$. *Furthermore, assume that (2.38) holds and there exists* $T \in [t_0, \infty)_{\mathbb{T}}$ *such that*

$$\int_{T}^{\infty} \left(\frac{1}{p(s)} \int_{T}^{s} g(u)\pi^{\beta}(u)\Delta u \right)^{\frac{1}{\gamma}} \Delta s = \infty. \tag{3.7}$$

Then every solution of (1.1) oscillates.

Proof. Suppose to the contrary and assume that y is a nonoscillatory solution of equation (1.1). Without loss of generality we may assume that $y(t) > 0$, $y(\tau(t)) > 0$, and $y(\delta(t)) > 0$ for $t \geq T$ (where T is chosen large enough so the conclusions of Lemmas 2.2 and 3.1 hold). We consider only this case, because the proof when $y(t) < 0$ is similar. From Remark 3.1, there are two possible cases: (3.1) and (3.2). First, we consider (3.1). In this case we proceed as in the proof of Theorem 2.6 and define $u(t)$ as in (2.9) to get a contradiction with (2.38). Now, we consider (3.2). Proceed as in the proof of Lemma 3.1 to get the inequality (3.3) where $x(t)$ satisfies (3.5) for $t \geq T$. From (3.5), since $x^{[1]}(t) < 0$, we have for $s \geq t \geq T$ that $-x^{[1]}(s) \geq$ $\geq -x^{[1]}(t)$, or

$$p(s)(-x^{\Delta}(s))^{\gamma} \geq p(t)(-x^{\Delta}(t))^{\gamma},$$

and hence

$$-x^{\Delta}(s) \geq \left(\frac{1}{p(s)}\right)^{\frac{1}{\gamma}} \left(p(t)(-x^{\Delta}(t))^{\gamma}\right)^{\frac{1}{\gamma}}.$$

Integrating from $t(\geq T)$ to u $(\geq t)$ and letting $u \to \infty$, yields

$$x(t) > -x(\infty) + x(t) \geq \left(p(t)(-x^{\Delta}(t))^{\gamma}\right)^{\frac{1}{\gamma}} \int_{t}^{\infty} \left(\frac{1}{p(s)}\right)^{\frac{1}{\gamma}} \Delta s = p^{\frac{1}{\gamma}}(t)(-x^{\Delta}(t)\pi(t) \quad \text{for} \quad t \geq T.$$

From this, since $p^{\frac{1}{\gamma}}(t)(-x^{\Delta}(t)$ is decreasing, we have

$$x(t) \geq p^{\frac{1}{\gamma}}(T)(-x^{\Delta}(T)\pi(t) = c\pi(t) \quad \text{for} \quad t \geq T, \tag{3.8}$$

where $c = p^{\frac{1}{\gamma}}(T)(-x^{\Delta}(T) > 0$. Using (3.8) in (3.3), we get

$$(p(t)\left(x^{\Delta}(t)\right)^{\gamma})^{\Delta} + g(t)c^{\beta}\pi^{\beta}(t) \leq 0 \quad \text{for} \quad t \geq T.$$

Integrating the last inequality from T to t, we have

$$-p(t)\left(x^{\Delta}(t)\right)^{\gamma} \geq -p(T)\left(x^{\Delta}(T)\right)^{\gamma} + c^{\beta} \int_{T}^{t} g(s)\pi^{\beta}(s)\Delta s \geq c^{\beta} \int_{T}^{t} g(s)\pi^{\beta}(s)\Delta s,$$

or

$$-x^{\Delta}(t) \geq c^{\frac{\beta}{\gamma}} \left(\frac{1}{p(t)} \int_{T}^{t} g(s)\pi^{\beta}(s)\Delta s \right)^{\frac{1}{\gamma}}.$$

Integrating from T to t, we obtain

$$\infty > x(t_1) > x(t_1) - x(t) \geq c^{\frac{\beta}{\gamma}} \int_{T}^{t} \left(\frac{1}{p(s)} \int_{T}^{s} g(u)\pi^{\beta}(u)\Delta u \right)^{\frac{1}{\gamma}} \Delta s,$$

which contradicts (3.7). This completes the proof.

Remark 3.2. Note the difference between the inequality (2.4) when (1.2) holds, and the inequality (3.3) when (1.3) holds.

Example 3.1. Assume that $\mathbb{T} = \mathbb{R}$ and consider the neutral equation

$$\left(t^2 \left(y(t) + (1 - \frac{1}{t}) y(\lambda t) \right)' \right)' + \frac{\kappa t^2}{\alpha(t)} y(\frac{\lambda}{2} t) = 0, \quad t \in [1, \infty)_{\mathbb{R}}, \tag{3.9}$$

where $\tau(t) = \lambda t > \delta(t) = \dfrac{\lambda}{2} t$ and $\alpha(t) = P(\delta(t), T)/P(t, T) > 0$ for any $T \geq 1$. Here $\gamma = \beta = 1$, $0 < \lambda < 1$, and

$$p(t) = t^2, \quad r(t) = \left(1 - \frac{1}{t} \right), \quad \text{and} \quad q(t) = \kappa t^2, \quad \text{where} \quad \kappa > 0.$$

This gives (noting $\eta^{\sigma}(t) = 1$) that

$$A(t) = P(t)\alpha(t) = \alpha(t) q(t)(1 - r(\delta(t)) = \frac{2\kappa t}{\lambda}, \quad g(t) = 2\kappa t,$$

and

$$\pi(t) := \int_{t}^{\infty} \left(\frac{1}{p(s)} \right)^{\frac{1}{\gamma}} \Delta s = \int_{t}^{\infty} \frac{1}{s^2} ds = \frac{1}{t}.$$

It is easy to see that the assumptions $(h_1)-(h_3)$, and (1.3) hold, since

$$\int_{1}^{\infty} \left(\frac{1}{p(s)} \right)^{\frac{1}{\gamma}} \Delta s = \int_{1}^{\infty} \frac{1}{s^2} ds \leq \infty. \tag{3.10}$$

To apply Theorem 3.1, it remains to discuss (2.38) and (3.7). First, we discuss (2.38). It is clear that (2.38) is satisfied since

$$\int_{t_0}^{\infty} A(s)\Delta s = \int_{t_0}^{\infty} A(s) ds = \int_{1}^{\infty} \frac{2\kappa s}{\lambda} ds = \infty.$$

It remains to discuss the condition (3.7). Note

$$\int_{T}^{\infty}\left(\frac{1}{p(s)}\int_{T}^{s} g(u)\pi^{\beta}(u)\Delta u\right)^{\frac{1}{\gamma}}\Delta s = 2\kappa\int_{1}^{\infty}\left(\frac{1}{s^2}\int_{1}^{s} s\frac{1}{s}\Delta u\right) ds = \kappa\int_{1}^{\infty}\left(\frac{1}{s^2}(s-1)\right) ds = \infty.$$

Then by Theorem 3.1, every solution of (3.9) oscillates. Note that none of the results established in [1, 2, 3, 7, 8, 11 – 13, 15, 16] can be applied on (3.9), since (1.2) does not hold (see (3.10)).

Remark 3.3. In Theorem 3.1, we used the condition (2.38) to get a contradiction if the (3.1) holds. Also we can use the conditions (2.41), (2.42), (2.43) and (2.44) to get a counteraction. For the case when (3.2) holds we proceed as in the proof of Theorem 3.1 to get a contradiction with (3.7). So that the following results can similarly be stated. There are, however, no new principles involved.

Theorem 3.2. *Assume that* $(h_1)-(h_3)$, *(1.3),* $\tau^{\Delta}(t) \geq 0$ *and* $r^{\Delta}(t) \geq 0$ *hold. Furthermore, assume that (2.41) holds and there exists* $T \in [t_0, \infty)_{\mathbb{T}}$ *such that (3.7) holds. Then every solution of (1.1) oscillates.*

Theorem 3.3. *Assume that* $(h_1)-(h_3)$, *(1.3) hold,* $\tau^{\Delta}(t) \geq 0$ *and* $r^{\Delta}(t) \geq 0$. *Furthermore, assume that (2.42) holds and there exists* $T \in [t_0, \infty)_{\mathbb{T}}$ *such that (3.7) holds. Then every solution of (1.1) oscillates.*

Theorem 3.4. *Assume that* $(h_1)-(h_3)$, *(1.3) hold,* $\tau^{\Delta}(t) \geq 0$ *and* $r^{\Delta}(t) \geq 0$. *Furthermore, assume that (2.43) holds and there exists* $T \in [t_0, \infty)_{\mathbb{T}}$ *such that (3.7) holds. Then every solution of (1.1) oscillates.*

Theorem 3.5. *Assume that* $(h_1)-(h_3)$, *(1.3) hold,* $\tau^{\Delta}(t) \geq 0$ *and* $r^{\Delta}(t) \geq 0$. *Furthermore, assume that (2.44) holds and there exists* $T \in [t_0, \infty)_{\mathbb{T}}$ *such that (3.7) holds. Then every solution of (1.1) oscillates.*

Remark 3.4. We note that the results in Theorems 3.1 – 3.5 are valid only when $\delta(t) \leq \tau(t) \leq$ $\leq t$. So it would be interesting to consider the case when this condition is not satisfied and find new oscillation criteria when (1.3) holds. Also it would be interesting to find new conditions different from the condition (3.7).

1. *Agarwal R. P., O'Regan D., Saker S. H.* Oscillation criteria for second-order nonlinear neutral delay dynamic equations // J. Math. Anal. and Appl. — 2004. — **300**. — P. 203 – 217.
2. *Agarwal R. P., O'Regan D., Saker S. H.* Oscillation results for second-order nonlinear neutral delay dynamic equations on time scales // Appl. Anal. — 2007. — **86**. — P. 1 – 17.
3. *Agarwal R. P., O'Regan D., Saker S. H.* Oscillation theorems for second-order nonlinear neutral delay dynamic equations on time scales // Acta math. sinica. — 2008. — **24**. — P. 1409 – 1432.
4. *Bohner M., Peterson A.* Dynamic equations on time scales // Introduction with Applications. — Boston: Birkhäuser, 2001.
5. *Bohner M., Peterson A.* Advances in dynamic equations on time scales. — Boston: Birkhäuser, 2003.
6. *Hilger S.* Analysis on measure chains-a unified approach to continuous and discrete calculus // Results Math. — 1990. — **18**. — P. 18 – 56.
7. *Mathsen R. M., Wang Q., Wu H.* Oscillation for neutral dynamic functional equations on time scales // J. Different. Equat. and Appl. — 2004. — **10**. — P. 651 – 659.

8. *Şahiner Y.* Oscillation of second-order neutral delay and mixed-type dynamic equations on time scales // Adv. Difference Equat. — 2006. — **2006**. — P. 1–9.

9. *Kac V., Cheung P.* Quantum calculus. — New York: Springer, 2001.

10. *Kelley W., Peterson A.* Difference equations // Introduction with Applications. — Second ed. — San Diego: Harcourt/Acad. Press, 2001.

11. *Saker S. H.* Oscillation of second-order nonlinear neutral delay dynamic equations on time scales // J. Comput. Appl. Math. — 2005. — **177**. — P. 375–387.

12. *Saker S. H.* Oscillation of second-order neutral delay dynamic equations of Emden-Fowler type // Dynam. Syst. Appl. — 2006. — **15**. — P. 629–644.

13. *Saker S. H.* Oscillation criteria for a certain class of second-order neutral delay dynamic equations // Dynamics Cont. Discrete Impulsive Syst. Ser. B: Appl. and Algorithms (to appear).

14. *Spedding V.* Taming Nature's Numbers // New Sci. — 2003. — P. 28–31.

15. *Tirpathy A. K.* Some oscillation results for second order nonlinear dynamic equations of neutral type // Nonlinear Anal. — 2009, doi: 10.1016/j. na.2009.02046.

16. *Wu H.-Wu, Zhuang R. K., Mathsen R. M.* Oscillation criteria for second-order nonlinear neutral variable delay dynamic equations // Appl. Math. — 2006. — **178**. — P. 231–331.

Received 21.03.09