DOI: 10.3842/nosc.v28i1.1492

UDC 517.9

ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF NONLINEAR SECOND-ORDER DIFFERENTIAL EQUATIONS THAT ARE IMPLICIT IN THE HIGHEST DERIVATIVE

АСИМПТОТИКА РОЗВ'ЯЗКІВ НЕЛІНІЙНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ ДРУГОГО ПОРЯДКУ, НЕ РОЗВ'ЯЗАНИХ ЩОДО СТАРШОЇ ПОХІДНОЇ

Viacheslav Yevtukhov, Liliia Koltsova

I. I. Mechnikov Odesa National University
Vsevoloda Zmiienka St., 2, Odesa, 65082, Ukraine
e-mail: evtukhov@onu.edu.ua
koltsova.liliya@gmail.com, corresponding author

We establish conditions for the existence and asymptotic representations as $t \to +\infty$ of monotonic solutions of nonlinear second-order differential equations unsolvable regarding the highest derivative.

Встановлено умови існування та асимптотичні зображення при $t \to +\infty$ монотонних розв'язків нелінійних диференціальних рівнянь другого порядку, не розв'язаних щодо старшої похідної.

1. Problem statement and formulation of the main result. A second-order ordinary differential equation of the form is considered:

$$F(t, y, y', y'') = \sum_{k=1}^{n} p_k(t) y^{\alpha_k} |y'|^{\beta_k} |y''|^{\gamma_k} = 0, \tag{1.1}$$

where $n \in \mathbb{N},\ n \geq 2,\ \alpha_k,\ \beta_k,\ \gamma_k \in \mathbb{R},\ \sum_{k=1}^{n} |\gamma_k| \neq 0,\ p_k \in \mathrm{C}([a; +\infty), a > 0; \mathbb{R}),\ k = \overline{1, n}$, i $p_i(t) \neq 0,\ i = \overline{1, s}$, for some $2 \leq s \leq n$.

For Equation (1.1), the question of the existence and asymptotic behavior as $t \to +\infty$ of solutions $y(t)$, which are indefinitely extendable to the right (r-solutions), is studied.

Earlier in [1], a similar question regarding the asymptotics of solutions of the equation of the form (1.1) was considered in the case when $\sum_{k=1}^{n} |\gamma_k| = 0$, that is, when Equation (1.1) is a first-order differential equation.

The main result of this work is obtained under the assumption of existence of a function $v \in \mathrm{C}^2([t_1; +\infty), t_1 > a; \mathbb{R})$, which satisfies the following conditions:

(A) $v(t) > 0,\ v''(t) \neq 0$ on $[t_1; +\infty),\ v(+\infty)$ is either equal to 0 or $+\infty$;

(B) $\lim_{t \to +\infty} \dfrac{v''(t) v(t)}{(v'(t))^2} = \mu,\ 0 \neq \mu \in \mathbb{R}$;

(C) $\lim_{t\to+\infty} \dfrac{p_i(t)v^{\alpha_i}(t)|v'(t)|^{\beta_i}|v''(t)|^{\gamma_i}}{p_1(t)v^{\alpha_1}(t)|v'(t)|^{\beta_1}|v''(t)|^{\gamma_1}} = c_i,\ 0 \neq c_i \in \mathbb{R},\ i = \overline{1,s},\ \sum_{i=1}^{s} \gamma_i c_i \neq 0,$ $\lim_{t\to+\infty} \dfrac{p_j(t)v^{\alpha_j}(t)|v'(t)|^{\beta_j}|v''(t)|^{\gamma_j}}{p_1(t)v^{\alpha_1}(t)|v'(t)|^{\beta_1}|v''(t)|^{\gamma_1}} = 0,\ j = \overline{s+1,n}.$

The following statement holds.

Theorem 1.1. *Let there exist a function* $v \in \mathrm{C}^2([t_1;+\infty), t_1 > a;\mathbb{R})$ *that satisfies conditions (A) – (C). Then, for the existence of* r*-solutions* $y(t)$ *of differential equation (1.1), which satisfy the asymptotic representations*

$$y^{(k)}(t) = v^{(k)}(t)(1+o(1)), \quad t\to+\infty, \quad k=\overline{0,2}, \tag{1.2}$$

it is necessary, and if the roots $\lambda_1,\ \lambda_2$ *of the algebraic equation*

$$\lambda^2 + \left(1 + \frac{\mu\sum_{i=1}^{s}(\beta_i+\gamma_i)c_i}{\sum_{i=1}^{s}\gamma_i c_i}\right)\lambda + \frac{\mu\sum_{i=1}^{s}(\alpha_i+\beta_i+\gamma_i)c_i}{\sum_{i=1}^{s}\gamma_i c_i} = 0 \tag{1.3}$$

have the property $\operatorname{Re}\lambda_k \neq 0,\ k=1,2,$ *then it is also sufficient that*

$$\sum_{i=1}^{s} c_i = 0. \tag{1.4}$$

Moreover, if $\operatorname{sign}(\operatorname{Re}\lambda_1) \neq \operatorname{sign}(\operatorname{Re}\lambda_2)$, *then there exists a one-parameter family of r-solutions with asymptotic representations (1.2), and if* $\operatorname{sign}(\operatorname{Re}\lambda_1) = \operatorname{sign}(\operatorname{Re}\lambda_2) \neq \operatorname{sign}(v'(t))$ *in some neighborhood of* $+\infty$, *then there exists a two-parameter family of* r*-solutions with asymptotic representations (1.2).*

2. Auxiliary statements. To prove Theorem 1.1, some auxiliary statements will be needed.

Lemma 2.1 [2]. *Let the system*

$$x_k' = \alpha(t)(f_k(x_1,x_2) + r_k(t,x_1,x_2)), \quad k=1,2, \tag{2.1}$$

where $(t,x_1,x_2) \in D_3,\ D_3 = [t_0;+\infty)\times H,\ H = [-h_1;h_1]\times[-h_2;h_2],\ t_0\in\mathbb{R},\ h_1,h_2>0,$ *satisfies the following conditions:*

(1) $\alpha \in \mathrm{C}([t_0;+\infty);\mathbb{R}),\ \alpha(t)\neq 0,\ \displaystyle\int_{t_0}^{+\infty}|\alpha(t)|\,dt = +\infty;$

(2) $f_k \in \mathrm{C}^{2\,2}_{x_1x_2}(H;\mathbb{R}),\ f_k(0,0)=0,\ k=1,2;$

(3) $r_k \in \mathrm{C}^{0\,1\,1}_{tx_1x_2}(D_3;\mathbb{R})$, *and*

$$\lim_{t\to+\infty}\sup_{(x_1,x_2)\in H}\left(\left|\frac{\partial r_k(t,x_1,x_2)}{\partial x_1}\right| + \left|\frac{\partial r_k(t,x_1,x_2)}{\partial x_2}\right| + |r_k(t,x_1,x_2)|\right) = 0, \quad k=1,2;$$

(4) *the roots* $\lambda_1,\ \lambda_2$ *of the algebraic equation*

$$\lambda^2 - (K_{11}+K_{22})\lambda + K_{11}K_{22} - K_{12}K_{21} = 0,$$

where $K_{ij} = \dfrac{\partial f_i(0;0)}{\partial x_j},\ i,j=1,2,$ *satisfy the condition* $\operatorname{Re}\lambda_k \neq 0,\ k=1,2.$

Then there exists a non-empty set of o-solutions of system (2.1)

$$\Omega = \big\{(x_1(t), x_2(t)) \in \mathrm{C}^1\big([t_1; +\infty),\ t_1 \ge t_0; \mathbb{R}^2\big)\colon x_1(+\infty) = x_2(+\infty) = 0\big\}.$$

Moreover, if either $\operatorname{sign}(\operatorname{Re}\lambda_1\alpha(t)) = -1$ *or* $\operatorname{sign}(\operatorname{Re}\lambda_2\alpha(t)) = -1,$ *then* Ω *is a one-parameter family of o-solutions, and if* $\operatorname{sign}(\operatorname{Re}\lambda_k\alpha(t)) = -1,\ k = 1, 2,$ *then* Ω *is a two-parameter family of* o*-solutions.*

Lemma 2.2. *Let the relation*

$$\Phi(t, x_1, x_2, x_3) = 0, \tag{2.2}$$

where $(t, x_1, x_2, x_3) \in D,\ D = D_0 \times [-h_3; h_3],\ D_0 = [a; +\infty) \times [-h_1; h_1] \times [-h_2; h_2],\ a \in \mathbb{R},$ $h_k > 0,\ k = 1, 2, 3,$ *satisfy the following conditions:*

(1) $\Phi,\ \Phi'_{x_1},\ \Phi'_{x_2},\ \Phi''_{x_3^2} \in \mathrm{C}(D; \mathbb{R})$*;*

(2) $\lim_{t\to+\infty} \Phi(t, 0, 0, 0) = 0$*;*

(3) $\lim_{t\to+\infty} \Phi'_{x_3}(t, 0, 0, 0) = A_1 \ne 0$*;*

(4) $\sup_D \Big|\Phi''_{x_3^2}(t, x_1, x_2, x_3)\Big| = A_2 < +\infty.$

Then, in some domain $D_1 = D_{01} \times [-h_3^*; h_3^*]$, *where* $D_{01} = [t_0; +\infty) \times [-h_1^*; h_1^*] \times [-h_2^*; h_2^*]$, t_0 *and* $h_k^*,\ k = 1, 2, 3$ *satisfy the inequalities* $t_0 \ge a,\ 0 < h_k^* \le h_k,\ \dfrac{4A_2h_3^*}{|A_1|} < 1,$ *the relation (2.2) defines a unique continuous function* $x_3 = x_3(t, x_1, x_2)$ *on the set* D_{01}, *such that* $\Phi(t, x_1, x_2, x_3(t, x_1, x_2)) \equiv 0,\ \dfrac{\partial x_3}{\partial x_1}, \dfrac{\partial x_3}{\partial x_2} \in \mathrm{C}(D_{01}; \mathbb{R}),\ x_3(+\infty, 0, 0) = 0$ *and*

$$x_3(t, 0, 0) \sim -\frac{\Phi(t, 0, 0, 0)}{\Phi'_{x_3}(t, 0, 0, 0)}, \quad t \to +\infty. \tag{2.3}$$

Proof. Since the function $\Phi(t, x_1, x_2, x_3)$ is differentiable with respect to the variable x_3 for $(t, x_1, x_2) \in D_0$, then, according to Taylor's formula with a remainder term in the Peano form, Equation (2.2) can be written as

$$\Phi(t, x_1, x_2, x_3) = \Phi(t, x_1, x_2, 0) + \Phi'_{x_3}(t, x_1, x_2, 0)x_3 + R(t, x_1, x_2, x_3) = 0, \tag{2.4}$$

where $R(t, x_1, x_2, x_3) = r(t, x_1, x_2, x_3)x_3$, and $r(t, x_1, x_2, x_3) \to 0$ as $x_3 \to 0$ for any point $(t, x_1, x_2) \in D_0$.

From (2.4), it follows that $x_3(t, x_1, x_2)$ is an implicit function determined by the relation

$$x_3(t, x_1, x_2) = \frac{-\Phi(t, x_1, x_2, 0) - R(t, x_1, x_2, x_3(t, x_1, x_2))}{\Phi'_{x_3}(t, x_1, x_2, 0)}. \tag{2.5}$$

Since

$$R(t, x_1, x_2, x_3) = \Phi(t, x_1, x_2, x_3) - \Phi(t, x_1, x_2, 0) - \Phi'_{x_3}(t, x_1, x_2, 0)x_3,$$

then

$$R'_{x_3}(t, x_1, x_2, x_3) = \Phi'_{x_3}(t, x_1, x_2, x_3) - \Phi'_{x_3}(t, x_1, x_2, 0).$$

Let us consider and estimate the difference $\Phi'_{x_3}(t, x_1, x_2, x_3^2) - \Phi'_{x_3}\big(t, x_1, x_2, x_3^1\big)$ for $(t, x_1, x_2) \in D_0$ and fixed $x_3^1, x_3^2 \in [-h_3; h_3],\ x_3^1 < x_3^2$, applying the Lagrange theorem with respect to the variable x_3:

$$\Phi'_{x_3}(t, x_1, x_2, x_3^2) - \Phi'_{x_3}\big(t, x_1, x_2, x_3^1\big) = \Phi''_{x_3x_3}(t, x_1, x_2, x_3^*)\big(x_3^2 - x_3^1\big),$$

where $x_3^* \in \big(x_3^1; x_3^2\big)$.

It follows that

$$\sup_{D_0}\bigl|\Phi'_{x_3}\bigl(t,x_1,x_2,x_3^2\bigr)-\Phi'_{x_3}\bigl(t,x_1,x_2,x_3^1\bigr)\bigr| \le \sup_{D_0}\bigl|\Phi''_{x_3x_3}(t,x_1,x_2,x_3^*)\bigr|\bigl|x_3^2-x_3^1\bigr|$$
$$\le \sup_{D}\bigl|\Phi''_{x_3x_3}(t,x_1,x_2,x_3)\bigr|\bigl|x_3^2-x_3^1\bigr|$$
$$= A_2\bigl|x_3^2-x_3^1\bigr|.$$

Setting $x_3^1=0,\ x_3^2=x_3,$ for any fixed $x_3\in[-h_3;h_3]$ we obtain

$$\sup_{D_0}\bigl|R'_{x_3}(t,x_1,x_2,x_3)\bigr| \le A_2|x_3|. \tag{2.6}$$

Similarly, we consider and estimate the difference $R(t,x_1,x_2,x_3^2)-R\bigl(t,x_1,x_2,x_3^1\bigr)$ for $(t,x_1,x_2)\in D_0$ and fixed $x_3^k\in[-h_3;h_3],\ k=1,2,$ applying the Lagrange theorem with respect to the variable x_3:

$$R(t,x_1,x_2,x_3^2)-R\bigl(t,x_1,x_2,x_3^1\bigr)=R'_{x_3}(t,x_1,x_2,x_3^{**})\bigl(x_3^2-x_3^1\bigr),$$

where $x_3^{**}\in\bigl(x_3^1;x_3^2\bigr)$.

Considering (2.6), we obtain that

$$\sup_{D_0}\bigl|R(t,x_1,x_2,x_3^2)-R\bigl(t,x_1,x_2,x_3^1\bigr)\bigr| \le \sup_{D_0}\bigl|R'_{x_3}(t,x_1,x_2,x_3)\bigr|\bigl|x_3^2-x_3^1\bigr| \le A_2\bigl|x_3^2-x_3^1\bigr|^2.$$

Setting $x_3^1=0,\ x_3^2=x_3,$ for any fixed $x_3\in[-h_3;h_3]$, we obtain

$$\sup_{D_0}|R(t,x_1,x_2,x_3)| \le A_2|x_3|^2. \tag{2.7}$$

By virtue of (2.7) and conditions (2), (3) of Lemma 2.2, there exists a domain $D_{01}=[t_0;+\infty)\times[-h_1^*;h_1^*]\times[-h_2^*;h_2^*],\ D_{01}\subset D_0,$ where $t_0\ge a,\ 0<h_k^*\le h_k,\ k=1,2,$ for which:

(1) $\sup_{D_{01}}|\Phi(t,x_1,x_2,0)| \le \dfrac{h_3^*|A_1|}{4}$, where $h_3^*\in(0;h_3]$ and $\dfrac{4A_2h_3^*}{|A_1|}<1$;

(2) $\inf_{D_{01}}\bigl|\Phi'_{x_3}(t,x_1,x_2,0)\bigr| > \dfrac{|A_1|}{2}$;

(3) $\sup_{D_{01}}|R(t,x_1,x_2,x_3)| \le \sup_{D_0}|R(t,x_1,x_2,x_3)| \le A_2|x_3|^2$.

Consider the Banach space B of bounded continuous functions $x_3(t,x_1,x_2)$ on D_{01} with the norm $\|x_3\|=\sup_{D_{01}}|x_3(t,x_1,x_2)|$.

In the Banach space B, consider the subspace $B_1\subset B$ of functions $x_3\in B$ for which $\|x_3\|\le h_3^*$ and define on B_1 the operator

$$T(t,x_1,x_2,x_3(t,x_1,x_2)) \equiv \frac{-\Phi(t,x_1,x_2,0)-R(t,x_1,x_2,x_3(t,x_1,x_2))}{\Phi'_{x_3}(t,x_1,x_2,0)}. \tag{2.8}$$

We will show, by using the contraction mapping principle, that the operator T has a fixed point $x_3\in B_1$, i.e., $T(t,x_1,x_2,x_3(t,x_1,x_2))=x_3(t,x_1,x_2)$.

(1) We will prove that if $x_3(t,x_1,x_2)\in B_1$, then $T(t,x_1,x_2,x_3(t,x_1,x_2))\in B_1$.

Since $x_3\in \mathrm{C}(D_{01};\mathbb{R})$, due to the structure of the operator, $T\in \mathrm{C}(D_{01};\mathbb{R})$.

From $\|x_3(t,x_1,x_2)\| \le h_3^*$, it follows that

$$\|T(t,x_1,x_2,x_3(t,x_1,x_2))\|$$
$$= \left\| \frac{-\Phi(t,x_1,x_2,0) - R(t,x_1,x_2,x_3(t,x_1,x_2))}{\Phi'_{x_3}(t,x_1,x_2,0)} \right\|$$
$$\le \frac{1}{\inf\limits_{D_{01}}\left|\Phi'_{x_3}(t,x_1,x_2,x_3)\right|}\left(\sup_{D_{01}}|\Phi(t,x_1,x_2,0)| + \sup_{D_{01}}|R(t,x_1,x_2,x_3(t,x_1,x_2))|\right)$$
$$\le \frac{2}{|A_1|}\left(\frac{h_3^*|A_1|}{4} + A_2|x_3|^2\right) \le \frac{h_3^*}{2} + \frac{h_3^*}{2} \le h_3^*.$$

(2) Let us check the contraction condition.
Choose arbitrary $x_3^1(t,x_1,x_2),\ x_3^2(t,x_1,x_2) \in B_1$, then

$$\left\|T\big(t,x_1,x_2,x_3^2(t,x_1,x_2)\big) - T\big(t,x_1,x_2,x_3^1(t,x_1,x_2)\big)\right\|$$
$$= \left\| \frac{R\big(t,x_1,x_2,x_3^2(t,x_1,x_2)\big) - R\big(t,x_1,x_2,x_3^1(t,x_1,x_2)\big)}{\Phi'_{x_3}(t,x_1,x_2,0)} \right\|$$
$$\le \frac{A_2}{\inf\limits_{D_{01}}\left|\Phi'_{x_3}(t,x_1,x_2,0)\right|}\left\|x_3^2(t,x_1,x_2) - x_3^1(t,x_1,x_2)\right\|^2$$
$$\le \frac{2A_2}{|A_1|}\left(\left\|x_3^2(t,x_1,x_2)\right\| + \left\|x_3^1(t,x_1,x_2)\right\|\right)\left\|x_3^2(t,x_1,x_2) - x_3^1(t,x_1,x_2)\right\|$$
$$\le \frac{4A_2h_3^*}{|A_1|}\left\|x_3^2(t,x_1,x_2) - x_3^1(t,x_1,x_2)\right\|.$$

We have demonstrated that the operator T maps the space B_1 into itself and is a contraction operator. Therefore, by the contraction mapping principle, there exists a unique function $x_3 = x_3(t,x_1,x_2) \in B_1$ such that $x_3(t,x_1,x_2) = T(t,x_1,x_2,x_3(t,x_1,x_2))$. By virtue of (2.8), this continuous function on the set D_{01} is the unique solution to Equation (2.5), satisfying the condition $\|x_3\| \le h_3^*$. Considering this condition and since $\Phi \in \mathrm{C}(D;\mathbb{R})$, by the local implicit function theorem, we can assert that $x_3, \dfrac{\partial x_3}{\partial x_1}, \dfrac{\partial x_3}{\partial x_2} \in \mathrm{C}(D_{01};\mathbb{R})$.

Let us prove that the function $x_3(t,x_1,x_2)$ has property (2.3) when $x_1 = 0,\ x_2 = 0$.

The function $x_3(t,x_1,x_2)$ satisfies equation (2.4), which, by setting $x_1 = 0,\ x_2 = 0$, can be written as

$$\Phi(t,0,0,0) + \Phi'_{x_3}(t,0,0,0)x_3(t,0,0) + r(t,0,0,x_3(t,0,0))x_3(t,0,0) \equiv 0.$$

From this, considering that $\Phi'_{x_3}(+\infty,0,0,0) = A_1 \ne 0$, we get

$$x_3(t,0,0)\left(1 + \frac{r(t,0,0,x_3(t,0,0))}{\Phi'_{x_3}(t,0,0,0)}\right) = -\frac{\Phi(t,0,0,0)}{\Phi'_{x_3}(t,0,0,0)}.$$

From last equation, property (2.3) follows.
The lemma is proved.

3. Proof of the main theorem. ***Proof of Theorem 1.1.*** *Necessity.* Let $y \in \mathrm{C}^2([t_1; +\infty); \mathbb{R})$ be a solution of differential equation (1.1) of the form (1.2), where $v(t)$ is a function with properties (A) – (C). Then, from differential equation (1.1), it follows that

$$\sum_{k=1}^{n} p_k(v(t))^{\alpha_k} |v'(t)|^{\beta_k} \big|v''(t)\big|^{\gamma_k} (1 + o(1)) = 0, \quad t \to +\infty. \tag{3.1}$$

Dividing Equation (3.1) by $p_1(t)(v(t))^{\alpha_1}|v'(t)|^{\beta_1}|v''(t)|^{\gamma_1}$ and taking into account condition (C), we get:

$$\sum_{i=1}^{s} c_i (1 + o(1)) = 0, \quad t \to +\infty,$$

which is possible only if condition (1.4) is satisfied.

Sufficiency. Let there exist a function $v(t)$ that satisfies conditions (A) – (C) and condition (1.4). Then, according to condition (C)

$$\frac{p_i(t)(v(t))^{\alpha_i}|v'(t)|^{\beta_i}|v''(t)|^{\gamma_i}}{p_1(t)(v(t))^{\alpha_1}|v'(t)|^{\beta_1}|v''(t)|^{\gamma_1}} = c_i + \varepsilon_i(t), \quad i = \overline{1, s},$$

$$\frac{p_j(t)(v(t))^{\alpha_j}|v'(t)|^{\beta_j}|v''(t)|^{\gamma_j}}{p_1(t)(v(t))^{\alpha_1}|v'(t)|^{\beta_1}|v''(t)|^{\gamma_1}} = \varepsilon_j(t), \quad j = \overline{s+1, n},$$

where $\varepsilon_k(t) = o(1),\ k = \overline{1, n},$ при $t \to +\infty$.

Let us show that, in this case, Equation (1.1) has a solution $y(t)$ of the form (1.2) and clarifies the question regarding the number of such solutions.

Substituting into (1.1)

$$y = v(t)(1 + x_1), \tag{3.2}$$

$$y' = v'(t)(1 + x_2), \tag{3.3}$$

$$y'' = v''(t)(1 + x_3), \tag{3.4}$$

we obtain the equation

$$\sum_{k=1}^{n} p_k(v(t))^{\alpha_k}(1 + x_1)^{\alpha_k} \big|v'(t)\big|^{\beta_k}(1 + x_2)^{\beta_k} \big|v''(t)\big|^{\gamma_k}(1 + x_3)^{\gamma_k} = 0. \tag{3.5}$$

After dividing by $p_1(t)(v(t))^{\alpha_1}|v'(t)|^{\beta_1}|v''(t)|^{\gamma_1}$, Equation (3.5) takes the form

$$\Phi(t, x_1, x_2, x_3) = 0, \tag{3.6}$$

where

$$\Phi(t, x_1, x_2, x_3) = \sum_{i=1}^{s} c_i(1 + x_1)^{\alpha_i}(1 + x_2)^{\beta_i}(1 + x_3)^{\gamma_i} + \sum_{k=1}^{n} \varepsilon_k(t)(1 + x_1)^{\alpha_k}(1 + x_2)^{\beta_k}(1 + x_3)^{\gamma_k}.$$

Consider $\Phi(t,x_1,x_2,x_3)$ on the set $D = D_0\times[-h_3;h_3]$, $D_0 = [a;+\infty)\times[-h_1;h_1]\times[-h_2;h_2]$. Due to its structure, $\Phi,\Phi'_{x_1},\Phi'_{x_2},\Phi''_{x_3^2} \in \mathrm{C}(D;\mathbb{R})$, and according to conditions (C) and (1.4), it has the following properties:

$$\lim_{t\to+\infty}\Phi(t,0,0,0)=0,\qquad \lim_{t\to+\infty}\Phi'_{x_3}(t,0,0,0)=\sum_{i=1}^{s}\gamma_i c_i\neq 0,\qquad \sup_D\Big|\Phi''_{x_3^2}(t,x_1,x_2,x_3)\Big|<+\infty.$$

Thus, for the function $\Phi(t,x_1,x_2,x_3)$, the conditions of Lemma 2.2 are satisfied.

Then, in some domain $D_1 = D_{01}\times[-h_3^*;h_3^*]$, where

$$D_{01}=[t_0;+\infty)\times[-h_1^*;h_1^*]\times[-h_2^*;h_2^*],\quad t_0\geq a,\quad 0<h_k^*\leq h_k,\quad k=1,2,3,$$

and

$$\frac{4h_3^*\sup\limits_D\Big|\Phi''_{x_3^2}(t,x_1,x_2,x_3)\Big|}{\Big|\sum_{i=1}^{s}\gamma_i c_i\Big|}<1,$$

Equation (3.6) defines a unique continuous function $x_3 = x_3(t,x_1,x_2)$ on the set D_{01}, such that $\Phi(t,x_1,x_2,x_3(t,x_1,x_2))\equiv 0$, $(x_3)'_{x_k}\in\mathrm{C}(D_{01};\mathbb{R})$, $k=1,2$, $x_3(+\infty,0,0)=0$ and

$$x_3(t,0,0)\sim-\frac{\sum_{i=1}^{s}c_i+\sum_{k=1}^{n}\varepsilon_k(t)}{\sum_{i=1}^{s}\gamma_i c_i+\sum_{k=1}^{n}\gamma_k\varepsilon_k(t)},\quad t\to+\infty. \tag{3.7}$$

Since

$$\frac{\partial x_3(t,x_1,x_2)}{\partial x_1}=-\frac{\Phi'_{x_1}(t,x_1,x_2,x_3)}{\Phi'_{x_3}(t,x_1,x_2,x_3)},\qquad \frac{\partial x_3(t,x_1,x_2)}{\partial x_2}=-\frac{\Phi'_{x_2}(t,x_1,x_2,x_3)}{\Phi'_{x_3}(t,x_1,x_2,x_3)},$$

we also have, as $t\to+\infty$,

$$\frac{\partial x_3(t,0,0)}{\partial x_1}=-\frac{\sum_{i=1}^{s}\alpha_i c_i(1+x_3(t,0,0))^{\gamma_i}+\sum_{k=1}^{n}\alpha_k\varepsilon_k(t)(1+x_3(t,0,0))^{\gamma_k}}{\sum_{i=1}^{s}\gamma_i c_i(1+x_3(t,0,0))^{\gamma_i-1}+\sum_{k=1}^{n}\gamma_k\varepsilon_k(t)(1+x_3(t,0,0))^{\gamma_k-1}}, \tag{3.8}$$

$$\frac{\partial x_3(t,0,0)}{\partial x_2}=-\frac{\sum_{i=1}^{s}\beta_i c_i(1+x_3(t,0,0))^{\gamma_i}+\sum_{k=1}^{n}\beta_k\varepsilon_k(t)(1+x_3(t,0,0))^{\gamma_k}}{\sum_{i=1}^{s}\gamma_i c_i(1+x_3(t,0,0))^{\gamma_i-1}+\sum_{k=1}^{n}\gamma_k\varepsilon_k(t)(1+x_3(t,0,0))^{\gamma_k-1}}. \tag{3.9}$$

Substituting the function $x_3(t,x_1,x_2)$ into (3.4) instead of x_3, and taking into account relations (3.2) – (3.4), we obtain the following system of differential equations for finding x_1 and x_2:

$$\begin{cases} x_1'=\dfrac{v'}{v}\,(-x_1+x_2),\\[2mm] x_2'=\dfrac{v''}{v'}\,(-x_2+x_3(t,x_1,x_2)).\end{cases} \tag{3.10}$$

Expanding the function $x_3(t,x_1,x_2)$ in terms of x_1 and x_2 for $t\in[t_0;+\infty)$ by using the Maclaurin series, the system (3.10) can be written as

$$\begin{cases} x_1'=\dfrac{v'}{v}\,(-x_1+x_2),\\[2mm] x_2'=\dfrac{v''}{v'}\left(x_3(t,0,0)+\dfrac{\partial x_3}{\partial x_1}\,(t,0,0)x_1+\left(-1+\dfrac{\partial x_3}{\partial x_2}(t,0,0)\right)x_2+r(t,x_1,x_2)\right),\end{cases} \tag{3.11}$$

where
$$r(t,x_1,x_2)=x_3(t,x_1,x_2)-x_3(t,0,0)-\frac{\partial x_3}{\partial x_1}(t,0,0)x_1-\frac{\partial x_3}{\partial x_2}(t,0,0)x_2,$$
and it is evident that $r(t,0,0)\equiv 0$.

Since, according to condition (B),
$$\frac{v''(t)}{v'(t)}=\frac{v'(t)}{v(t)}(\mu+\delta(t)),\quad 0\neq\mu\in\mathbb{R},\quad \delta(t)\in \mathrm{C}([t_0;+\infty);\mathbb{R}),$$
and as $t\to+\infty,\ \delta(t)=o(1),$ the system (3.11) can be written as
$$\begin{cases} x_1'=\dfrac{v'}{v}\,(-x_1+x_2),\\ x_2'=\dfrac{v'}{v}\,\Big((\mu+\delta(t))x_3(t,0,0)+(\mu+\delta(t))\dfrac{\partial x_3}{\partial x_1}(t,0,0)x_1\\ \qquad +(\mu+\delta(t))\Big(-1+\dfrac{\partial x_3}{\partial x_2}(t,0,0)\Big)x_2+(\mu+\delta(t))r(t,x_1,x_2)\Big). \end{cases} \tag{3.12}$$

The system (3.12) is a system of the form (2.1), where:

(1) $\alpha(t)=\dfrac{v'(t)}{v(t)}$. By virtue of condition (A) $\displaystyle\int_{t_0}^{+\infty}\frac{v'(t)}{v(t)}\,dt=\pm\infty$;

(2) $f_i(x_1,x_2)=K_{i1}x_1+K_{i2}x_2,\ f_i\in \mathrm{C}^{2\,2}_{x_1x_2}([-h_1^*;h_1^*]\times[-h_2^*;h_2^*];\mathbb{R}),\ f_i(0,0)=0,\ i=1,2,$ where $K_{11}=-1,\ K_{12}=1$, considering conditions (1.4), (3.7)–(3.9):
$$K_{21}=\lim_{t\to+\infty}(\mu+\delta(t))\,\frac{\partial x_3}{\partial x_1}(t,0,0)$$
$$=\lim_{t\to+\infty}(\mu+\delta(t))\left(-\frac{\sum_{i=1}^{s}\alpha_ic_i(1+x_3(t,0,0))^{\gamma_i}+\sum_{k=1}^{n}\alpha_k\varepsilon_k(t)(1+x_3(t,0,0))^{\gamma_k}}{\sum_{i=1}^{s}\gamma_ic_i(1+x_3(t,0,0))^{\gamma_i-1}+\sum_{k=1}^{n}\gamma_k\varepsilon_k(t)(1+x_3(t,0,0))^{\gamma_k-1}}\right)$$
$$=-\frac{\mu\sum_{i=1}^{s}\alpha_ic_i\left(1-\dfrac{\sum_{i=1}^{s}c_i}{\sum_{i=1}^{s}\gamma_ic_i}\right)^{\gamma_i}}{\sum_{i=1}^{s}\gamma_ic_i\left(1-\dfrac{\sum_{i=1}^{s}c_i}{\sum_{i=1}^{s}\gamma_ic_i}\right)^{\gamma_i-1}}=-\frac{\mu\sum_{i=1}^{s}\alpha_ic_i}{\sum_{i=1}^{s}\gamma_ic_i},$$
$$K_{22}=\lim_{t\to+\infty}(\mu+\delta(t))\Big(-1+\frac{\partial x_3}{\partial x_2}(t,0,0)\Big)$$
$$=\lim_{t\to+\infty}(\mu+\delta(t))\left(-1-\frac{\sum_{i=1}^{s}\beta_ic_i(1+x_3(t,0,0))^{\gamma_i}+\sum_{k=1}^{n}\beta_k\varepsilon_k(t)(1+x_3(t,0,0))^{\gamma_k}}{\sum_{i=1}^{s}\gamma_ic_i(1+x_3(t,0,0))^{\gamma_i-1}+\sum_{k=1}^{n}\gamma_k\varepsilon_k(t)(1+x_3(t,0,0))^{\gamma_k-1}}\right)$$
$$=-\mu\left(1+\frac{\sum_{i=1}^{s}\beta_ic_i\left(1-\dfrac{\sum_{i=1}^{s}c_i}{\sum_{i=1}^{s}\gamma_ic_i}\right)^{\gamma_i}}{\sum_{i=1}^{s}\gamma_ic_i\left(1-\dfrac{\sum_{i=1}^{s}c_i}{\sum_{i=1}^{s}\gamma_ic_i}\right)^{\gamma_i-1}}\right)$$

$$= -\mu\left(1+\frac{\sum_{i=1}^{s}\beta_i c_i}{\sum_{i=1}^{s}\gamma_i c_i}\right) = -\frac{\mu\sum_{i=1}^{s}(\beta_i+\gamma_i)c_i}{\sum_{i=1}^{s}\gamma_i c_i};$$

(3)

$$r_1(t,x_1,x_2)=0,$$

$$r_2(t,x_1,x_2) = (\mu+\delta(t))(x_3(t,0,0)+r(t,x_1,x_2))$$

$$= (\mu+\delta(t))\left(x_3(t,x_1,x_2)-\frac{\partial x_3}{\partial x_1}(t,0,0)x_1-\frac{\partial x_3}{\partial x_2}(t,0,0)x_2\right),$$

$r_k \in \mathrm{C}^{01\,1}_{tx_1x_2}(D_{01};\mathbb{R})$, and $\sup_{x_1,x_2}\left(\left|\frac{\partial r_k}{\partial x_1}\right|+\left|\frac{\partial r_k}{\partial x_2}\right|+|r_k|\right)=o(1)$ as $t\to+\infty,\ k=1,2;$

(4) the algebraic equation

$$\lambda^2-(K_{11}+K_{22})\lambda+K_{11}K_{22}-K_{12}K_{21}=0$$

has the form (1.3), whose roots have the property $\operatorname{Re}\lambda_k\neq 0,\ k=1,2.$

Thus, on the set D_{01}, all the conditions of Lemma 2.1 are satisfied for system (3.12). Then, system (3.12) has a non-empty set of o-solutions

$$\Omega=\big\{(x_1(t),x_2(t))\in\mathrm{C}^1\big([t_1;+\infty],t_1\geq t_0;\mathbb{R}^2\big)\colon x_1(+\infty)=x_2(+\infty)=0\big\}.$$

Each such solution of system (3.12), by virtue of the properties of the function $x_3(t,x_1,x_2)$ and the substitutions (3.2) – (3.4), corresponds to an r-solution of system (1.1) of the form (1.2).

Moreover, if $\operatorname{sign}(\operatorname{Re}\lambda_1)\neq\operatorname{sign}(\operatorname{Re}\lambda_2)$, then there exists a one-parameter family of r-solutions of the form (1.2). If, in some neighborhood of $+\infty$, $\operatorname{sign}(\operatorname{Re}\lambda_1)=\operatorname{sign}(\operatorname{Re}\lambda_2)\neq\operatorname{sign}(v'(t))$, then there exists a two-parameter family of r-solutions of the form (1.2).

The Theorem is proved.

4. Conclusions. In this work, for the first time, a necessary condition for the existence of r-solutions of the form (1.2) for the differential equation (1.1) has been obtained and, under certain conditions, a sufficient condition has also been established. Additionally, the question of the number of such solutions has been clarified.

On behalf of all authors, the corresponding author states that there is no conflict of interest. All necessary data are included into the paper. All authors contributed equally to this work. The authors declare no special funding of this work.

References

1. L. Koltsova, A. Kostin, *The asymptotic behavior of solutions of monotone type of first-order nonlinear ordinary differential equations, unresolved for the derivative*, Mem. Differ. Equat. Math. Phys., № 57, 51 – 74 (2012).
2. A. V. Kostin, *Asymptotics of proper solutions of ordinary differential equations*, Dr. Diss., Kyiv, Institute of Mathematics, National Academy of Sciences of Ukraine (1991).

Received 03.12.24