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AUTONOMOUS PERIODIC BOUNDARY-VALUE PROBLEMS WITH SWITCHINGS AT NONFIXED POINTS IN TIME

АВТОНОМНІ ПЕРІОДИЧНІ КРАЙОВІ ЗАДАЧІ З ПЕРЕМИКАННЯМИ У НЕФІКСОВАНІ МОМЕНТИ ЧАСУ

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We analyze the relationship between boundary-value problems with impulsive action at fixed points of time and boundary-value problems with switchings at fixed and nonfixed points of time. We find the constructive conditions of solvability and a scheme for construction of solutions of a nonlinear periodic boundary-value problem with switchings at nonfixed points of time. By using the Adomian decomposition method, we obtain the solvability conditions and construct a new iterative technique for finding solutions of a weakly nonlinear periodic boundary-value problem with switchings at nonfixed points of time. In addition, we obtain constructive conditions for convergence of the iterative scheme to the solution of the weakly nonlinear boundary-value problem as well as the switchings points. The obtained iterative scheme is applied to find approximations to the periodic solution of the equation with switchings at nonfixed points of time, which models a nonisothermal chemical reaction.

Проаналізовано зв'язок крайових задач із імпульсним збуренням у фіксовані моменти часу та крайових задач із перемиканнями у фіксовані й нефіксовані моменти часу. Одержано конструктивні умови розв'язності та схему побудови розв'язків нелінійної періодичної крайової задачі з перемиканнями у нефіксовані моменти часу. З використанням методу декомпозиції Адомяна отримано умови розв'язності та побудовано нову ітераційну техніку для знаходження розв'язків слабконелінійної періодичної крайової задачі з перемиканнями у нефіксовані моменти часу. Отримано конструктивні умови збіжності побудованої ітераційної схеми до розв'язку слабконелінійної крайової задачі, а також моментів перемикання. Здобуту ітераційну схему застосовано для знаходження наближень до періодичного розв'язку рівняння з перемиканнями у нефіксовані моменти часу, яке моделює неізотермічну хімічну реакцію.

1. Preliminaries. The study of impulsively perturbed systems of ordinary differential equations is traditional for the Kyiv school of nonlinear oscillations, in particular, for research at the Institute of Mathematics of the National Academy of Sciences of Ukraine. It was initiated in 1937 by M. M. Krylov and M. M. Bogolubov with the study of a clock mechanism in which the damping of oscillations caused by friction was compensated by periodic jolts of the anchor [1]. In addition, in 1937, the impulsively perturbed mechanical and electrical vibrations were discussed in the monograph [2].

Before 1967, the studies of the theory of impulsive boundary-value problems was characterized by a descriptive character, limited to the discussion of examples of impulsive processes in mechanics, electronics, physiology; in addition, the study of the theory of impulsive boundary-value problems was focused on linear systems [3 – 5].

The study of the theory of impulsive boundary-value problems was initiated in Kharkiv by A. D. Myshkis in connection with the question of engineer B. V. Abramov about the influence of periodic shocks on the stability of the engine. A. D. Myshkis suggested it as a topic for a thesis by V. D. Milman [6 – 8]. It is worth mentioning the research of A. D. Myshkis on systems with jolts [6, 9, 10] and equations with switching [11, 12], which were generalizations of T. Vogel's perturbation systems [13]. The research of M. M. Krylov and M. M. Bogolyubov was continued in the work of A. D. Mishkis and A. M. Samoilenko [14]. The constructive theory of systems of ordinary differential equations with impulsive action is gaining intensive development after the publication of monographs by A. M. Samoilenko and M. O. Perestyuk [15, 16], as well as its English translation [17].

In the works [15 – 30] were found necessary and sufficient conditions for the existence of solutions of impulsively perturbed boundary-value problems for systems of ordinary differential equations in various critical and noncritical cases, as well as the construction of the Green's operator of the Cauchy problem and the Green's operator of a periodic and almost periodic [30 – 32] boundary-value problem with impulsive action [22, 28, 33]. A characteristic feature of all these publications is the use of either a nondegenerate or a two-point impulsive action with rectangular matrices; the latter problems were studied by R. Conti [34 – 36]. They also generalized the two-point problems with square matrices studied in [4, 5], and in the case of nondegeneracy of these matrices considered in [37].

In the works of A. M. Samoilenko, M. O. Perestyuk and O. A. Boichuk, the necessary and sufficient conditions for the existence of solutions of Noether boundary-value problems for systems of ordinary differential equations with nondegenerate impulsive action at fixed points in time [20, 21, 26, 38 – 41]. In the monographs by A. M. Samoilenko and M. O. Perestiuk [15, 16] systems of ordinary differential equations with impulse action at nonfixed points in time, namely, with impulsive action on the hyperplane, are also investigated.

The relevance of the study of impulsively perturbed ordinary differential equations is due to their numerous applications, in particular in the theory of nonlinear oscillations [2], control theory [42 – 47], in mechanics, biology [3, 48] and radio engineering [49, 50], the theory of motion stability [6, 16, 51], nuclear physics, financial mathematics, logistics, where the time of rapid changes can be neglected in comparison with the time of slow changes, which are also determined by a system of ordinary differential equations. An example of such changes is the phenomenon of a spherical membrane, the oscillations of which outside the moments of flapping are modeled by a system of ordinary differential equations, and the shocks themselves at the moments of the overfloating depend on the state of the membrane and are determined separately. On the other hand, the questions of existence and construction of solutions to boundary-value problems with impulsive action occupy an honorable place in the qualitative theory of functional-differential equations [20, 52], ordinary differential equations [18, 40, 49, 53], in particular in the problems of optimal control [16], the theory of stochastic differential equations [54 – 58], the theory of differential equations with multivalued and discontinuous right-hand side, and differential equations with inclusions [59].

Development of the theory of linear systems with nondegenerate

$$\det(I_n + S_i) \neq 0$$

impulsive action at fixed points in time

$$z(t) \in \mathbb{C}^1\{[a,b] \setminus \{\tau_i\}_I\}, \quad a := \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_p < b$$

is significantly related to the studies of A. M. Samoilenko and M. O. Perestyuk of linear systems [15, 16, 28, 29, 40]

$$\frac{dz}{dt} = A(t)z + f(t), \quad t \neq \tau_i, \qquad \Delta z(\tau_i) = S_i\, z(\tau_i - 0) + a_i, \quad a_i \in \mathbb{R}^n, \tag{1}$$

where $A(t)$ is a continuous $(n \times n)$-dimensional matrix, $f(t)$ is a continuous vector function, S_i is an $(n \times n)$-dimensional matrix. Furthermore, the development of the theory of linear Noether $(m \neq n)$ boundary-value problems with nondegenerate impulsive action at fixed points in time

$$\frac{dz}{dt} = A(t)z + f(t), \quad t \neq \tau_i, \qquad \Delta z(\tau_i) = S_i\, z(\tau_i - 0) + a_i, \quad \ell z(\cdot) = 0 \tag{2}$$

was initiated by A. M. Samoilenko and O. A. Boichuk [20, 33, 40, 41, 60, 61]. Here,

$$\ell z(\cdot) : \mathbb{C}^1\{[a,b] \setminus \{\tau_i\}_I\} \to \mathbb{R}^m$$

is a linear bounded vector functional.

If for some i matrices

$$I_n + S_i, \quad i = 1, 2, \ldots, p,$$

are degenerate, there is a degenerate impulsive action [62, 63]. In comparison with the problems with nondegenerate impulsive action, the rank of the normal fundamental matrix $X(t)$ of the Cauchy problem with degenerate impulsive action becomes less than n in the intervals $[\tau_q, \tau_{q+1}[, \ldots, [\tau_p, b] \subset [a, b]$, after the point τ_q of the first degenerate impulsive action. After the point τ_q of the first degenerate impulsive action

$$\det(I_n + S_q) = 0$$

the rank of the normal fundamental matrix $X(t)$ of the Cauchy problem with degenerate impulsive action does not increase [62, 64 – 68].

In the case of degenerate impulsive action, the solution of the homogeneous part of the system (1) is represented by a normal fundamental matrix. In turn, the scheme [20, 33, 40, 41, 60, 61] can not be used to construct the solution of the inhomogeneous system (1). To construct solutions of linear Noether $(m \neq n)$ boundary-value problems with degenerate impulse action at fixed points in time for a inhomogeneous system (1) the necessary and sufficient conditions for existence and the construction of the Green's operator of the Cauchy problems as well as the Green's operator of a Noether boundary-value problem with impulsive action in critical and noncritical cases are obtained [62, 63, 66 – 69].

The boundary-value problems with switching at fixed points in time are a special case of boundary-value problems with impulsive action [70, 71].

The continuity of the sought solution of a nonlinear boundary-value problem with switches at nonfixed points in time in the modelling of nonisothermal chemical reactions [72, 73] follows from the chemical meaning of continuous changes in the unknowns.

A generalization of boundary-value problems, both with degenerate and nondegenerate impulsive action, is the problem of finding the conditions of existence and constructing solutions

$$z(t) \in \mathbb{C}^1\{[a,b] \setminus \{\tau_i\}_I\}, \quad j = 1, 2, \ldots, n,$$

of the system of ordinary differential equations

$$\frac{dz}{dt} = A_i(t)z + f(t), \quad t \neq \tau_i, \tag{3}$$

that satisfy the boundary condition [74 – 81]

$$\mathcal{L}z(\cdot) = \alpha, \quad \alpha \in \mathbb{R}^m, \quad i = 1, 2, \ldots, p, \tag{4}$$

where $\mathcal{L}z(\cdot)$ is a linear bounded vector functional of the form

$$\mathcal{L}z(\cdot) = \sum_{i=0}^{p} \ell_i \, z(\cdot),$$

and

$$\ell_i z(\cdot) \colon \mathbb{C}^1[\tau_i, \tau_{i+1}] \times \ldots \times \mathbb{C}^1[\tau_i, \tau_{i+1}] \to \mathbb{R}^m, \quad i = 0, 1, 2, \ldots, p-1, \quad \tau_0 = a,$$

$$\ell_p z(\cdot) \colon \mathbb{C}^1[\tau_p, b] \times \ldots \times \mathbb{C}^1[\tau_p, b] \to \mathbb{R}^m$$

are linear bounded functionals. Here, $A(t)$ is a continuous $(n \times n)$-dimensional matrix, $f(t)$ is a continuous vector function.

The problem (3), (4) is a generalization of boundary-value problems with both degenerate [62, 66 – 68] and nondegenerate impulsive action [15, 16, 28, 29, 40]. If the impulsive action (4) is determined by the functional

$$\ell_0 z(\cdot) = \begin{bmatrix} N_1 z(\tau_1 - 0) \\ O_{n\times 1} \\ \ldots\ldots \\ O_{n\times 1} \\ O_{n\times 1} \end{bmatrix}, \qquad \ell_1 z(\cdot) = \begin{bmatrix} M_1 z(\tau_1 + 0) \\ N_2 z(\tau_2 - 0) \\ \ldots\ldots \\ O_{n\times 1} \\ O_{n\times 1} \end{bmatrix}, \ldots,$$

which act respectively from the spaces

$$\mathbb{C}^1[a,\tau_1]\times\ldots\times\mathbb{C}^1[a,\tau_1],\quad \mathbb{C}^1[\tau_1,\tau_2]\times\ldots\times\mathbb{C}^1[\tau_1,\tau_2],\ldots$$

to the space $\mathbb{R}^{pk}$, as well as to the functionals

$$\ell_{p-1}z(\cdot)=\begin{bmatrix} O_{n\times 1}\\ O_{n\times 1}\\ \ldots\ldots\\ M_{p-1}z(\tau_{p-1}+0)\\ N_p z(\tau_p-0)\end{bmatrix},\qquad \ell_p z(\cdot)=\begin{bmatrix} O_{n\times 1}\\ O_{n\times 1}\\ \ldots\ldots\\ O_{n\times 1}\\ M_p z(\tau_p+0)\end{bmatrix},$$

that act respectively from the spaces

$$\mathbb{C}^1[\tau_{p-1},\tau_p]\times\ldots\times\mathbb{C}^1[\tau_{p-1},\tau_p],\quad \mathbb{C}^1[\tau_p,b]\times\ldots\times\mathbb{C}^1[\tau_p,b]$$

to the space $\mathbb{R}^{pk}$, we get the problem which considered in [34]. Here, $M_i,\ N_i$ are $(k\times n)$ matrices. In particular, for

$$\mathbb{R}(N_i)=\mathbb{R}(M_i),\quad \mathbb{N}(N_i)=\varnothing,$$

this problem studied in [22], in the the case $k=n$ it studied in [4, 5] and provided that the matrices are nondegenerate $M_i,\ N_i$ it studied in the [37]. Here, $\mathbb{R}(N_i),\ \mathbb{N}(N_i)$ are respectively the images and null spaces of matrices N_i. Further, provided that matrices are nondegenerate $N_i=-(I_n+S_i)$ and equalities $M_i=I_n$ is fulfilled we obtain the problem studied in [16, 20, 33, 82]. In particular, for $S_i=0$ we have the problem [83]. If for some i the matrices I_n+S_i are degenerate, there is a degenerate impulse action [62].

Thus, for the problems studied by S. Shvabik, A. M. Samoilenko, M. O. Perestyuk, and O. A. Boichuk, as well as for the problems with degenerate impulsive action, the functionals that determine the discontinuity of the integral curve at the points $\tau_1,\tau_2,\ldots$ use information about this curve only at these points. The functionals $\ell_0 z(\cdot)$ and $\ell_1 z(\cdot)$ determine the first $(i=1)$ discontinuity of the integral curve at the point τ_1, the functionals $\ell_1 z(\cdot)$ and $\ell_2 z(\cdot)$ define the second $(i=2)$ discontinuity of the integral curve at the point τ_2, then the functionals $\ell_{p-1}z(\cdot)$ and $\ell_p z(\cdot)$ define the last $(i=p)$ discontinuity of the integral curve at point τ_p. If the impulsive action (4) is determined by the functional

$$\ell_0 z(\cdot)=\begin{bmatrix}\ell_1^{(0)}z(\cdot)\\ \ell_2^{(0)}z(\cdot)\\ \ldots\\ \ell_{p-1}^{(0)}z(\cdot)\\ \ell_p^{(0)}z(\cdot)\end{bmatrix},\qquad \ell_1 z(\cdot)=\begin{bmatrix}\ell_1^{(1)}z(\cdot)\\ \ell_2^{(1)}z(\cdot)\\ \ldots\\ \ell_{p-1}^{(1)}z(\cdot)\\ \ell_p^{(1)}z(\cdot)\end{bmatrix},\qquad \ell_2 z(\cdot)=\begin{bmatrix}O_{n\times 1}\\ \ell_2^{(2)}z(\cdot)\\ \ldots\\ \ell_{p-1}^{(2)}z(\cdot)\\ \ell_p^{(2)}z(\cdot)\end{bmatrix},$$

that act respectively from spaces

$$\mathbb{C}^1[a,\tau_1]\times\ldots\times\mathbb{C}^1[a,\tau_1],\quad \mathbb{C}^1[\tau_1,\tau_2]\times\ldots\times\mathbb{C}^1[\tau_1,\tau_2],\ldots,$$

$$\mathbb{C}^1[\tau_2, \tau_3] \times \ldots \times \mathbb{C}^1[\tau_2, \tau_3], \ldots$$

to the space $\mathbb{R}^{(p+1)k}$, and the functionals

$$\ell_{p-1} z(\cdot) = \begin{bmatrix} O_{n\times 1} \\ O_{n\times 1} \\ \cdots \\ \ell_{p-1}^{(p-1)} z(\cdot) \\ \ell_p^{(p-1)} z(\cdot) \end{bmatrix}, \qquad \ell_p z(\cdot) = \begin{bmatrix} O_{n\times 1} \\ O_{n\times 1} \\ \cdots \\ O_{n\times 1} \\ \ell_p^{(p)} z(\cdot) \end{bmatrix},$$

that act respectively from spaces

$$\mathbb{C}^1[\tau_{p-1}, \tau_p] \times \ldots \times \mathbb{C}^1[\tau_{p-1}, \tau_p], \quad \mathbb{C}^1[\tau_p, b] \times \ldots \times \mathbb{C}^1[\tau_p, b]$$

to the space $\mathbb{R}^{(p+1)k}$, we obtain a problem with boundary conditions of the "interface conditions" type studied in [34, 67, 68, 75, 76, 84], where

$$\ell_i^{(0)} z(\cdot) : \mathbb{C}[a, \tau_1] \to \mathbb{R}^k, \ldots, \ell_i^{(i)} z(\cdot) : \mathbb{C}[\tau_i, \tau_{i+1}] \to \mathbb{R}^k, \quad i = 1, \ldots, p-1, \ldots,$$

$$\ell_p^{(0)} z(\cdot) : \mathbb{C}[a, \tau_1] \to R^k, \ldots, \ell_p^{(p)} z(\cdot) : \mathbb{C}[\tau_p, b] \to \mathbb{R}^k$$

are linear bounded functionals. Thus, for problems with boundary conditions of the "interface conditions" type, the discontinuity of the integral curve which extended from the interval $[a, \tau_1]$ to the interval $[\tau_1, \tau_2]$ is determined using the functionals $\ell_1^{(0)} z(\cdot)$ and $\ell_1^{(1)} z(\cdot)$. This functionals are defined in contrast to the problems studied by A. M. Samoilenko, M. O. Perestiuk, and S. Shvabik, and to the problems with degenerate impulsive action, on the whole length of these intervals, not only on their intersection. The discontinuity of the integral curve at extension from the interval $[a, \tau_1] \cup [\tau_1, \tau_2]$ to the interval $[\tau_2, \tau_3]$ is determined by using the functionals $\ell_2^{(0)} z(\cdot),\ \ell_2^{(1)} z(\cdot)$ and $\ell_2^{(2)} z(\cdot)$, defined on the interval $[a, \tau_3]$ except the points τ_1 and τ_2. Further, the discontinuity of the integral curve which extended from the interval

$$[a, \tau_1] \cup [\tau_1, \tau_2] \cup \ldots \cup [\tau_{p-1}, \tau_p]$$

to the interval $[\tau_p, b]$ is determined by the functional $\ell_p^{(0)} z(\cdot),\ \ell_p^{(1)} z(\cdot), \ldots, \ell_p^{(p)} z(\cdot)$. This functionals are defined on the interval $[a, b]$ except the points $\tau_1, \tau_2, \ldots, \tau_p$. A special case of problems of the form (3), (4) is a series of p unrelated boundary-value problems

$$\frac{dz}{dt} = A_i(t) z, \quad \ell_i z(\cdot) = \alpha_i,$$

defined on the segments

$$t \in [\tau_i, \tau_{i+1}], \quad i = 1, 2, \ldots, p,$$

where

$$\alpha_i \in R^{m_i}, \operatorname{col}(\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_p) \in R^m, \quad m_1 + m_2 + \ldots + m_p = m.$$

Finally, the boundary-value problem (3), (4) is a generalization of the traditional problem of finding smooth solutions of the system (3) that satisfying the linear boundary condition [60]. On the other hand, the boundary condition (4) is equivalent to the condition [85], and the boundary-value problem (3), (4) is a special case of boundary-value problems for functional differential equations [85]. In difference from the problems with nondegenerate impulsive action, as well as from the problems with boundary conditions of the “interface conditions” type [34, 67, 68, 75, 76, 84], the rank of any of the fundamental matrix $X(t)$ of the problem (3), (4) can be less than n on any interval $[a, \tau_1[, [\tau_1, \tau_2[, \ldots, [\tau_p, b] \subset [a, b],$ in particular, on the half-interval $[a, \tau_1[.$ An example of the latter case is any fundamental matrix $X(t)$ of the traditional problem [60] of finding smooth periodic solutions of the system (3) provided that the system (3) contains both periodic and nonperiodic solutions. Since the rank of any of the fundamental matrices $X(t)$ of the problem (3), (4) can be less than n on the half-interval $[a, \tau_1[,$ then the definition of the normal fundamental matrix $X(t)$ of the problem (3), (4).

A special case of boundary-value problems with impulsive action are boundary-value problems with switches at nonfixed points in time [86], which, in particular, arise in the modelling of nonisothermal chemical reactions [71 – 73]. In comparison with the classical results of A. M. Samoilenko and M. O. Perestyuk [15, 16, 28, 29, 40], and the results of A. M. Samoilenko and O. A. Boichuk [20, 33, 40, 41, 60, 61, 87] we will construct solutions of linear systems (1) in the class of continuous functions

$$z(\cdot) \in \mathbb{C}^1\{[0, T] \setminus \{\tau(\varepsilon)\}_I\} \cap \mathbb{C}[0, T]$$

with one unknown continuous switching $\tau(\varepsilon)$:

$$0 < \tau(\varepsilon) < T,$$

while the results of A. M. Samoilenko, M. O. Perestyuk and O. A. Boichuk were obtained in the class of, generally speaking, discontinuous functions

$$z(t) \in \mathbb{C}^1\{[a, b] \setminus \{\tau_i\}_I\}, \quad a := \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_p < b,$$

with an impulsive action at fixed points in time.

2. Conditions of solvability of a quasilinear periodic boundary-value problem with switchings at nonfixed points in time. We study the problem of constructing solutions [17, 20]

$$z(\cdot, \varepsilon) \in \mathbb{C}^1\{[0, T] \setminus \{\tau(\varepsilon)\}_I\} \cap \mathbb{C}[0, T], \quad z(t, \cdot) \in \mathbb{C}[0, \varepsilon_0]$$

of the autonomous boundary-value problem for the equation

$$z'(t, \varepsilon) = Az(t, \varepsilon) + \varepsilon Z(z(t, \varepsilon), \varepsilon), \quad \ell z(\cdot, \varepsilon) = 0, \tag{5}$$

which continuous at $t = \tau(\varepsilon)$. At the point $t = \tau(\varepsilon)$:

$$0 < \tau(\varepsilon) < T, \quad \tau(0) := \tau_0$$

the solution of the boundary-value problem (5) might have a limited discontinuity of first derivative [17, 20]. The solution of the boundary-value problem (5) is found in a small neighbourhood of the solution

$$z_0(t) \in \mathbb{C}\{[0, T] \setminus \{\tau_0\}_I\} \cap \mathbb{C}[0, T]$$

of the generating boundary-value problem

$$z_0'(t) = A\, z_0(t), \quad \ell z_0(\cdot) = 0. \tag{6}$$

At the point $t = \tau_0$ the solution of the boundary-value problem (6) might have a limited discontinuity of the derivative. Here, $A \in \mathbb{R}^{n\times n}$ is a constant matrix, $Z(z, \varepsilon)$ is a nonlinear vector function piecewise analytic in the unknown z in a small neighbourhood of the solution of the generating problem (6) and piecewise analytic in a small parameter ε on the interval $[0, \varepsilon_0]$. In addition,

$$\ell z(\cdot, \varepsilon) := \begin{pmatrix} z(0,\varepsilon) - z(T,\varepsilon) \\ z(\tau(\varepsilon)+0,\varepsilon) - z(\tau(\varepsilon)-0,\varepsilon) \end{pmatrix} = 0$$

and

$$\ell z_0(\cdot) := \begin{pmatrix} z_0(0) - z_0(T) \\ Z_0(\tau_0+0) - z_0(\tau_0-0) \end{pmatrix} = 0$$

are linear bounded vector functionals.

The autonomous boundary-value problem (5) with switchings continues the study of Noether boundary-value problems, including boundary-value problems with switchings at fixed points in time [72], as well as with an impulsive action at fixed points in time [17, 20, 22, 75 – 77, 81], and also the study of nonlinear autonomous boundary-value problems [20, 88, 89]. As is well known [88, 89], the autonomous boundary-value problem (5) differs significantly from similar nonautonomous boundary-value problems. Unlike the latter, the right end $T(\varepsilon)$ of the interval $[0, T(\varepsilon)]$, where the required solution of the nonlinear boundary-value problem for the system (5) without switchings is defined, is unknown and subject to determination in the process of solution construction. In the article [72] we studied the autonomous nonlinear boundary-value problem with switchings at fixed points in time, which is solvable on a fixed length interval. Thus, the purpose of this article is to study the autonomous nonlinear boundary-value problem (5) with switchings at nonfixed points in time on a fixed length interval. An example of the relevance of studying such a problem will be given below.

In the article [72] was formulated the autonomous nonlinear boundary-value problem with switchings at fixed points in time defined on a fixed length interval and was obtained conditions when this problem has no solutions except for the equilibrium positions. At the same time, the involvement of switchings at fixed points in time allows us to obtain solutions different from the equilibrium positions [72]. On the other hand, the involvement of switchings at fixed points in time, in general, deprives the autonomous nonlinear boundary-value problem with switchings at fixed points in time, defined on a fixed length interval, of continuous solutions.

Example 1. Let us demonstrate the fact that, in general, the inclusion of switchings at the fixed points in time θ, deprives the autonomous nonlinear boundary-value problem with switchings at fixed points in time of

$$z'(t,\varepsilon) = \omega(\varepsilon)\, A\, z(t,\varepsilon) + \varepsilon Z(z(t,\varepsilon),\varepsilon), \quad \ell z(\cdot,\varepsilon) = 0, \tag{7}$$

defined on a fixed length interval $[0, T]$ of continuous solutions

$$z(\cdot,\varepsilon) \in \mathbb{C}\{[0,T] \setminus \{\theta\}_I\} \cap \mathbb{C}[0,T], \quad z(t,\cdot) \in \mathbb{C}[0,\varepsilon_0].$$

Here,

$$A := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Z(z(t,\varepsilon),\varepsilon) := (1 + x(t,\varepsilon))\, e^{\frac{\varepsilon}{1+y(t,\varepsilon)}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and

$$z(t,\varepsilon) := \begin{cases} x(t,\varepsilon), & t \in [0,\theta[, \\ y(t,\varepsilon), & t \in [\theta, T], \end{cases} \qquad \omega(\varepsilon) := \begin{cases} 1+\varepsilon, & t \in [0,\theta[, \\ 1-\varepsilon, & t \in [\theta, T], \end{cases}$$

$$\ell z(\cdot,\varepsilon) := \begin{pmatrix} z(0,\varepsilon) - z(T,\varepsilon) \\ z(\theta+0,\varepsilon) - z(\theta-0,\varepsilon) \end{pmatrix}, \quad T := 2\pi.$$

The boundary-value problem (7) has an equilibrium position of the form

$$z(t,\varepsilon) = \left(-\varepsilon + 3\,\varepsilon^2 - \frac{21\,\varepsilon^3}{2} + \ldots\right) \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad t \in [0,\theta[.$$

In addition,

$$z(t,\varepsilon) = \left(\varepsilon - \varepsilon^2 + \frac{5\,\varepsilon^3}{2} + \ldots\right) \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad t \in [\theta, T],$$

are defined on a fixed length interval $[0,T]$ which devoid of continuity for any $0 < \theta < T = 2\pi$. Denote the matrix

$$Q := \ell X(\cdot) := \begin{pmatrix} X(0) & -X(T) \\ X(\tau_0) & -X(\tau_0) \end{pmatrix} \in \mathbb{R}^{2n \times 2n},$$

where

$$X(t) \colon X'(t) = A\,X(t), \quad X(0) = I_n$$

is the normal fundamental matrix of the generating boundary-value problem (6). Let us assume that, for the generating boundary-value problem (6), there is a critical case [17, 20, 75, 76]:

$$P_{Q_r^*} \neq 0.$$

The matrix $P_{Q_r^*}$ is formed from r linearly independent rows of the orthoprojector matrix

$$P_{Q^*} \colon \mathbb{R}^{2n} \to \mathbb{N}(Q^*).$$

The generating problem (6) has a r parametric family of solutions

$$z_0(t, c_0) = X_r(t) c_0, \quad c_0 \in \mathbb{R}^r.$$

Here, $X_r(t) := X(t) P_{Q_r}$ is a matrix formed from r linearly independent solutions of the generating problem (6). The matrix P_{Q_r} is formed from r linearly independent columns of the orthoprojector matrix

$$P_Q \colon \mathbb{R}^{2n} \to \mathbb{N}(Q).$$

Denote by $f(t)$ a linear continuous vector function on the interval $[0,T]$. The solution

$$w(t) = K[f(s);\theta](t) := \int_0^t X(t) X^{-1}(s)\, f(s)\, ds, \quad t \in [0,\theta],$$

$$w(t) = K[f(s);\theta](t) := \int_{2\pi}^{t} X(t)X^{-1}(s)\, f(s)\, ds, \quad t \in [\theta, T],$$

of Cauchy problem for the system

$$w'(t) = A\, w(t) + f(t) \tag{8}$$

satisfies the boundary condition

$$\ell w(\cdot) := \begin{pmatrix} w(0) - w(T) \\ w(\theta + 0) - w(\theta - 0) \end{pmatrix} = 0$$

in the case of

$$P_{Q_r^*}\, \ell K[f(s);\theta](\cdot) = 0, \quad \theta \in [0, T],$$

for an arbitrary fixed point $\theta \in [0, T]$ of switching.

The autonomous boundary-value problem for a system (5) with switchings at nonfixed points in time on a fixed length interval is significantly different from similar boundary-value problems with switchings at fixed points in time [72]. To illustrate this, consider the following example.

Example 2. Let us consider the problem of finding the solution

$$w(\cdot, \varepsilon) \in \mathbb{C}^1\{[0, T] \setminus \{\theta(\varepsilon)\}_I\} \cap \mathbb{C}[0, T], \quad w(t, \cdot) \in \mathbb{C}[0, \varepsilon_0]$$

of the boundary-value problem

$$w'(t, \varepsilon) = \omega^2(\varepsilon)\, A\, w(t, \varepsilon) + f(t), \quad \ell z(\cdot, \varepsilon) = 0, \tag{9}$$

which is continuous at $t = \theta(\varepsilon)$. The solution of the boundary-value problem (9) is found in the small neighbourhood of the solution

$$w_0(t) \in \mathbb{C}\{[0, T] \setminus \{\theta_0\}_I\} \cap \mathbb{C}[0, T], \quad \theta_0 := \theta(0), \quad \omega_0 := \omega(0)$$

of the generating boundary-value problem

$$w_0'(t) = \omega_0^2\, A\, w_0(t) + f(t), \quad A \in \mathbb{R}^{n \times n}, \quad \ell z_0(\cdot) = 0. \tag{10}$$

Denote by $U(t) \colon U'(t) = A\,U(t);\ U(0) = I_n$ is the normal fundamental matrix of the generating boundary-value problem (10), and the matrix

$$Q := \ell X(\cdot) := \begin{pmatrix} U(0) & -U(T) \\ U(\theta_0) & -U(\theta_0) \end{pmatrix} \in \mathbb{R}^{2n \times 2n}.$$

For the generating boundary-value problem (10) there is a critical case [17, 20, 75, 76]:

$$P_{Q_r^*} \neq 0.$$

The matrix $P_{Q_r^*}$ is formed from r linearly independent rows of the orthoprojector matrix

$$P_{Q^*} : \mathbb{R}^{2n} \to \mathbb{N}(Q^*).$$

The generating problem (10) has r parametric family of solutions

$$w_0(t,c_0) = U_r(t)\,c_0 + K[f(s);\theta_0](t), \quad c_0 \in \mathbb{R}^r.$$

Under the condition

$$\int_0^T H_r^*(s)\,f(s)\,ds = 0$$

the generating problem (10) has r parametric family of continuous T periodic solutions

$$w_0(t,c_0) = U_r(t)\,c_0 + G[f(s);\theta_0](t), \quad c_0 \in \mathbb{R}^r,$$

where

$$G[f(s);\theta_0](t) = K[f(s);\theta_0](t) - X(t)\,Q^+\ell K[f(s);\theta_0](\cdot)$$

is the Green's operator of the generating T periodic problem (10) and $H_r(t)$ is an $(n \times r)$-dimensional matrix formed from r linearly independent T periodic solutions of the system that conjugates [88, 90, 91] to the generating T periodic problem (10);

$$U_r(t) := U(t)\,P_{Q_r}.$$

The matrix P_{Q_r} is formed from r linearly independent columns of the orthoprojector matrix

$$P_Q : \mathbb{R}^{2n} \to \mathbb{N}(Q).$$

The solution of the boundary-value problem (9) with switchings is given in the form

$$w(t,\varepsilon) := w_0(t,c_0^*) + u_1(t,\varepsilon) + \ldots + u_k(t,\varepsilon) + \ldots,$$

$$\theta(\varepsilon) = \theta_0 + \xi_1(\varepsilon) + \xi_2(\varepsilon) + \ldots + \xi_k(\varepsilon) + \ldots.$$

The first approximation to the solution of the nonlinear periodic boundary-value problem (9) with switchings in the critical case

$$w_1(t,\varepsilon) = w_0(t,c_0) + v_1(t,\varepsilon), \quad \theta_1(\varepsilon) = \theta_0 + \xi_1(\varepsilon),$$

$$w_1(t,\varepsilon) = U(t)\,c_1(\varepsilon) + G\big[\big(\omega^2(\varepsilon) - \omega_0^2\big)w_0(s,c_0)\big](t)$$

determines the solution of the nonlinear periodic boundary-value problem of the first approximation

$$w_1'(t,\varepsilon) = \big(\omega^2(\varepsilon) - \omega_0^2\big)w_0(t,c_0), \quad \ell w_1(\cdot,\varepsilon) = 0.$$

The condition of solvability of the periodic boundary-value problem of the first approximation

$$P_{Q_d^*}\,\ell K\big[\big(\omega^2(\varepsilon) - \omega_0^2\big)w_0(s,c_0);\theta(\varepsilon)\big](\cdot) = 0$$

leads to the equation

$$F_0(c_0,\theta_0) := P_{Q_d^*}\,\ell K\big[\omega_0^2\,w_0(s,c_0);\theta_0\big](\cdot) = 0. \tag{11}$$

The equation (11) will be called the equation for the generating constants of the nonlinear periodic boundary-value problem (9) with switchings in the critical case. The solvability of the equation (11) is a necessary condition for the existence of a solution to the periodic boundary-value problem (9) with switchings. The function $\theta(\varepsilon)$ plays the role of an eigenfunction [92, 93] of the nonlinear periodic boundary-value problem (9) with switchings in the critical case, which ensures the solvability of this problem. The equation (11) is, in general, nonlinear, despite the fact that the periodic boundary-value problem (9) with switchings at nonfixed points in time on a fixed length interval is linear.

Let us assume that the equation for the generating constants (11) of the periodic boundary-value problem (9) with switchings has real roots. Fixing one of the real solutions

$$c_0^* \in \mathbb{R}^n, \quad \theta_0^* \in \mathbb{R}$$

of equation (11), we get the problem of constructing a solution to the periodic boundary-value problem (9) with switchings in a small neighbourhood of the solution

$$w_0(t, c_0^*) = U_r(t)\, c_0^* + G[f(s); \theta_0^*](t)$$

of the generating boundary-value problem (10). A traditional condition for the solvability of the boundary-value problem (9) with switchings in a small neighbourhood of the solution of the generating problem is the requirement [20, 90]

$$P_{B_0^*} P_{Q_r^*} = 0, \quad B_0 := F'_{\check{c}_0}(c_0^*, \theta_0^*) \in \mathbb{R}^{r\times(r+1)}, \quad \check{c}_0 := \begin{pmatrix} c_0 & \theta_0 \end{pmatrix}^*,$$

where

$$P_{B_0^*} : \mathbb{R}^r \to \mathbb{N}(B_0^*)$$

is an orthoprojector matrix [20, 90]. Indeed, generally speaking, the nonlinear equation (11) for a linear periodic boundary-value problem (9) with switchings is equivalent to

$$B_0\, \check{c}_0 = P_{Q_d^*}\, \ell K \big\{ \omega_0^2 G[f(s); \theta](t) \big\}(\cdot),$$

solvable under the condition $P_{B_0^*} P_{Q_r^*} = 0$.

Let us put for definiteness

$$A := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad f(t) := \begin{pmatrix} 0 \\ \cos 3t \end{pmatrix}, \quad \omega(\varepsilon) := \begin{cases} 1+\varepsilon, & t \in [0, \theta(\varepsilon)[, \\ 1-\varepsilon, & t \in [\theta(\varepsilon), T]. \end{cases}$$

The generating problem (10) has a family of solutions

$$w_0(t, c_0) = U_r(t)\, c_0 + K[f(s); \theta_0](t), \quad c_0 \in \mathbb{R}^r,$$

where

$$U_r(t) = U(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \quad t \in [0, 2\pi],$$

and

$$K[f(s); \theta_0](t) = \frac{1}{8} \begin{pmatrix} \cos t - \cos 3t \\ -\sin t + 3 \sin 3t \end{pmatrix}, \quad t \in [0, 2\pi].$$

Let us put

$$\theta_0^* := \frac{\pi}{2}, \quad c_0 := \begin{pmatrix} c_{0a} & c_{0b} \end{pmatrix}^*.$$

The equation for the generating constants (11) of the periodic boundary-value problem (9) with switchings

$$F_0(c_0, \theta_0) = \frac{1}{4}\begin{pmatrix} 8\,c_{0a}\pi + \pi \\ 8\,c_{0b}\pi - 4 \end{pmatrix} = 0$$

has a real solution

$$c^*_{0a} = -\frac{1}{8}, \quad c^*_{0b} = \frac{1}{2\pi}, \quad \theta^*_0 = \frac{\pi}{2},$$

where

$$B_0 = \frac{1}{2\pi}\begin{pmatrix} 4\pi^2 & 0 & -3\,\pi \\ 0 & 4\pi^2 & -4 \end{pmatrix}$$

is the full rank matrix. To construct solutions of the autonomous periodic boundary-value problem (9) with switchings the method of simple iteration [20] can be used. Applying the method of simple iterations. In the first step we obtain

$$w_1(t,\varepsilon) := w_0(t, c^*_0) + u_1(t,\varepsilon), \quad \theta_1(\varepsilon) = \theta^*_0 + \xi_1(\varepsilon), \quad \xi_1(\varepsilon) = \varepsilon\,\theta^*_0,$$

where

$$u_1(t,\varepsilon) = \frac{\varepsilon}{16\pi}\begin{pmatrix} 5\pi\cos t + 16\,t\cos t - 5\pi\cos 3t + 16\sin t \\ 16\cos t - 5\pi\sin t - 16\,t\sin t + 3\pi\sin 3t \end{pmatrix}, \quad t \in [0, \theta_1(\varepsilon)],$$

$$u_1(t,\varepsilon) = \frac{\varepsilon}{16\pi}\begin{pmatrix} 27\pi\cos t - 16\,t\cos t + 5\pi\cos 3t + 16\sin t \\ 16\cos t - 27\pi\sin t + 16\,t\sin t - 3\pi\sin 3t \end{pmatrix}, \quad t \in [\theta_1(\varepsilon), 2\pi].$$

In the second step we get

$$w_2(t,\varepsilon) := w_0(t, c^*_0) + u_1(t,\varepsilon) + u_2(t,\varepsilon), \quad \theta_2(\varepsilon) = \theta^*_0 + \xi_1(\varepsilon) + \xi_2(\varepsilon), \quad \xi_2(\varepsilon) = \varepsilon^2\,\theta^*_0,$$

where

$$u_2(t,\varepsilon) = \frac{\varepsilon^2}{96\,\pi}\begin{pmatrix} 45\,\pi\cos t + 240\,t\cos t - 36\,\pi\cos 3t + 336\sin t + 29\pi^2\sin t \\ 336\cos t + 29\,\pi^2\cos t - 60\,\pi\,t\cos t - 96t^2\cos t - 45\pi\sin t \end{pmatrix}$$

$$+ \frac{\varepsilon^2}{96\,\pi}\begin{pmatrix} -60\,\pi\,t\sin t - 96\,t^2\sin t \\ -240\,t\sin t + 36\pi\sin 3t \end{pmatrix}, \quad t \in [0, \theta_2(\varepsilon)],$$

$$u_2(t,\varepsilon) = \frac{\varepsilon^2}{96\,\pi}\begin{pmatrix} 33\,\pi\cos t - 144\,t\cos t - 36\,\pi\cos 3t + 336\sin t \\ 336\cos t - 235\,\pi^2\cos t + 324\pi\,t\cos t - 96\,t^2\cos t \end{pmatrix}$$

$$+ \frac{\varepsilon^2}{96\,\pi}\begin{pmatrix} -235\pi^2\sin t + 324\,t\sin t - 96\,t^2\sin t \\ -333\,\pi\sin t + 144\,t\sin t + 36\,\pi\sin 3t \end{pmatrix}, \quad t \in [\theta_2(\varepsilon), 2\pi].$$

In the third step we have

$$w_3(t,\varepsilon) := w_0(t, c^*_0) + u_1(t,\varepsilon) + u_2(t,\varepsilon) + u_3(t,\varepsilon),$$

$$\theta_3(\varepsilon) = \theta_0^* + \xi_1(\varepsilon) + \xi_2(\varepsilon) + \xi_3(\varepsilon), \quad \xi_3(\varepsilon) = \varepsilon^3\, \theta_0^*,$$

where

$$u_3(t,\varepsilon) = \frac{\varepsilon^3}{192\,\pi}\begin{pmatrix} 93\,\pi\cos t + 1536\,t\cos t + 116\,\pi^2 t\cos t - 120\,\pi t^2\cos t \\ -240\,\pi\cos t - 576\,t^2\cos t - 93\,\pi\sin t - 1536\,t\sin t \end{pmatrix}$$

$$+\ \frac{\varepsilon^3}{192\,\pi}\begin{pmatrix} -128\,t^3\cos t - 93\,\pi\cos 3t - 240\,\pi t\sin t - 576\,t^2\sin t \\ -116\,\pi^2 t\sin t + 120\,\pi t^2\sin t + 128 t^3\sin t + 99\,\pi\sin 3t \end{pmatrix}, \quad t \in [0, \theta_3(\varepsilon)],$$

$$u_3(t,\varepsilon) = \frac{\varepsilon^3}{192\,\pi}\begin{pmatrix} 2211\,\pi\cos t - 312\,\pi^3\cos t - 1152\,t\cos t + 940\,\pi^2 t\cos t \\ -1248\,\pi^2\cos t + 1008\,\pi t\cos t - 192\,t^2\cos t - 2211\,\pi\sin t \end{pmatrix}$$

$$+\ \frac{\varepsilon^3}{192\,\pi}\begin{pmatrix} -648\,\pi t^2\cos t + 128\,t^3\cos 3t - 1248\,\pi^2\sin t + 1008\,\pi t\sin t \\ 312\,\pi^3\sin t + 1152\,t\sin t - 940\,\pi^2 t\sin t + 648\,\pi t^2\sin t \end{pmatrix}$$

$$+\ \frac{\varepsilon^3}{192\,\pi}\begin{pmatrix} -192\,t^2\sin t \\ -128\,t^3\sin t - 99\,\pi\sin 3t \end{pmatrix}, \quad t \in [\theta_3(\varepsilon), 2\pi].$$

For the obtained approximations there are constants

$$0 < \gamma := 0.997 < 1, \quad 0 < \delta := 0.997 < 1,$$

for which the inequalities

$$\|u_1(t,\varepsilon)\| \le \gamma \|w_0(t, c_0^*)\|, \quad \|u_{k+1}(t,\varepsilon)\| \le \gamma \|u_k(t,\varepsilon)\|,$$

$$|\xi_1(\varepsilon)| \le \delta\, |\theta_0^*|, \quad |\xi_{k+1}(\varepsilon)| \le \delta\, |\xi_k(\varepsilon)|, \quad k = 0, 1, 2,$$

is hold. This fact indicates the practical convergence of the obtained approximations to the solution of the autonomous periodic boundary-value problem (9) with switchings in the interval

$$\varepsilon \in [0, \varepsilon_0], \quad 0 \le \varepsilon_0 \approx 0.292\,580.$$

The accuracy of the found approximations to the solution of the autonomous periodic boundary-value problem (9) with switchings is determined by the inequalities

$$\Delta_k(\varepsilon) := \big\| w_k'(t,\varepsilon) - \omega^2(\varepsilon)\, A\, w_k(t,\varepsilon) - f(t) \big\|, \quad k = 1, 2, 3.$$

In addition,

$$\Delta_1(0.1) \approx 0.022\,727, \quad \Delta_2(0.1) \approx 0.00\,609\,356, \quad \Delta_3(0.1) \approx 0.00\,258\,018,$$

$$\Delta_1(0.01) \approx 0.000\,236\,948, \quad \Delta_2(0.01) \approx 6.75\,706 \times 10^{-6}, \quad \Delta_3(0.01) \approx 2.75\,350 \times 10^{-7}.$$

Note that the approximations to the solution of the periodic boundary-value problem (9) with switchings satisfy the boundary condition.

The condition for the solvability of the autonomous nonlinear boundary-value problem (5) with switchings

$$P_{Q_r^*}\, \ell K[Z(z(s,\varepsilon),\varepsilon); \tau(\varepsilon)](\cdot) = 0$$

leads to the equation

$$F_0(c_0, \tau_0) := P_{Q_r^*}\, \ell K[Z(z_0(s, c_0), 0); \tau_0](\cdot) = 0. \tag{12}$$

The necessary conditions for the existence of a solution to the autonomous nonlinear boundary-value problem (5) with switchings in the critical case are defined by the following lemma.

Lemma. *Suppose that there is the critical case for the generating boundary-value problem (6). In this case, the generating problem (6) has an one-parameter family of solutions*

$$z_0(t, c_0) = X_r(t)\, c_0, \quad c_0 \in \mathbb{R}^r.$$

Suppose that an autonomous nonlinear boundary-value problem (5) with switchings at nonfixed points in time in the neighbourhood of the generating solution $z_0(t, c_0)$ *has the solution*

$$z(\cdot, \varepsilon) \in \mathbb{C}\{[0, T] \setminus \{\tau(\varepsilon)\}_I\} \cap \mathbb{C}[0, T], \quad z(t, \cdot) \in \mathbb{C}[0, \varepsilon_0].$$

Under these conditions the equality (12) holds.

The equation (11) will be further called the equation for the generating constants of the boundary-value problem (5) with switchings in the critical case. Let us assume that the equation for the generating constants (11) of the boundary-value problem (5) with switchings has real roots. Fixing one of the real solutions

$$c_0^* \in \mathbb{R}^r, \quad \tau_0^* \in \mathbb{R}$$

of the equation (11) we get the problem of constructing a solution of the nonlinear boundary-value problem (5) in a small neighbourhood of the solution

$$z_0(t, c_0^*) = X_r(t)\, c_0^*, \quad c_0^* \in \mathbb{R}^r,$$

of the generating boundary-value problem (6). The traditional condition for the solvability of a boundary-value problem (5) with switchings in a small neighbourhood of the solution of the generating problem is the requirement [20, 90]

$$P_{B_0^*} P_{Q_r^*} \neq 0, \quad B_0 := F'_{\check{c}_0}(c_0^*, \tau_0^*) \in \mathbb{R}^{r \times (r+1)}, \quad \check{c}_0 := \begin{pmatrix} c_0 & \tau_0 \end{pmatrix}^*, \tag{13}$$

where

$$P_{B_0^*} : \mathbb{R}^r \to \mathbb{N}(B_0^*)$$

is an orthoprojector matrix [20, 90].

The solution of the boundary-value problem (5) with switchings is given by

$$z(t, \varepsilon) := z_0(t, c_0^*) + u_1(t, \varepsilon) + \ldots + u_k(t, \varepsilon) + \ldots,$$

$$\tau(\varepsilon) = \tau_0^* + \xi_1(\varepsilon) + \xi_2(\varepsilon) + \ldots + \xi_k(\varepsilon) + \ldots.$$

The nonlinear vector function $Z(z(t, \varepsilon), \varepsilon)$ is analytical with respect to the unknown $z(t, \varepsilon)$ in a small neighbourhood of the solution of the generating boundary-value problem (6) and the constant τ_0^*. Therefore, in the given neighbourhood there is an expansion [94, 95]

$$Z(z(t, \varepsilon), \varepsilon) = Z_0(z_0(t, c_0^*), \varepsilon) + Z_1(z_0(t, c_0^*), u_1(s, \varepsilon), \varepsilon)$$
$$+ Z_2(z_0(t, c_0^*), u_1(s, \varepsilon), u_2(s, \varepsilon), \varepsilon) + \ldots.$$

The first approximation to the solution of the nonlinear periodic boundary-value problem (5) in the critical case

$$z_1(t,\varepsilon) = z_0(t,c_0^*) + u_1(t,\varepsilon), \quad \tau_1(\varepsilon) = \tau_0^* + \xi_1(\varepsilon),$$
$$u_1(t,\varepsilon) = X_r(t)c_1(\varepsilon) + \varepsilon\, G\big[Z_0\big(z_0(s,c_0^*), z_0'(s,c_0^*),\varepsilon\big);\tau_0^*\big](t)$$

determines the solution of the nonlinear periodic boundary-value problem of the first approximation

$$u_1'(t,\varepsilon) = Au_1(t,\varepsilon) + \varepsilon\, Z_0(z_0(t,c_0^*),\varepsilon), \quad \ell u_1(\cdot,\varepsilon) = 0.$$

The matrix $B_0,$ which is the key matrix in the study of the boundary-value problem (5) takes the form

$$B_0 = P_{Q_r^*}\,\ell K[\mathcal{A}_0(s)X_r(s);\ 1](\cdot),$$

where

$$\mathcal{A}_0(t) = \left.\frac{\partial Z(z(t,\varepsilon),\varepsilon)}{\partial z(t,\varepsilon)}\right|_{\substack{z(t,\varepsilon)=z_0(t,c_0^*),\\ \varepsilon=0}}$$

is an $(r \times r)$-dimensional matrix. The second approximation to the solution of the nonlinear periodic boundary-value problem (5) in the critical case

$$z_2(t,\varepsilon) := z_0(t,c_0^*) + u_1(t,\varepsilon) + u_2(t,\varepsilon), \quad \tau_2(\varepsilon) = \tau_0^* + \xi_1(\varepsilon) + \xi_2(\varepsilon),$$

determines the solution of the nonlinear periodic boundary-value problem of the second approximation

$$u_2'(t,\varepsilon) = Au_2(t,\varepsilon) + \varepsilon Z_1(z_0(t,c_0^*), u_1(t,\varepsilon),\varepsilon), \quad \ell u_2(\cdot,\varepsilon) = 0.$$

The condition of solvability of the boundary-value problem of the second approximation

$$F_1(c_1(\varepsilon),\xi_1(\varepsilon)) := P_{Q_d^*}\ell K\big[Z_1\big(z_0(s,c_0^*), u_1(s,\varepsilon), z_0'(s,c_0^*), u_1(s,\varepsilon),\varepsilon\big);\xi_1(\varepsilon)\big](\cdot) = 0$$

is a linear equation

$$F_1(c_1(\varepsilon),\xi_1(\varepsilon)) = B_0\,\check{c}_1(\varepsilon) + \gamma_1(\varepsilon) = 0, \quad \check{c}_1(\varepsilon) := \begin{pmatrix} c_1(\varepsilon) & \xi_1(\varepsilon)\end{pmatrix}^*,$$

that at least uniquely solvable in the case (13). Here,

$$\gamma_1(\varepsilon) := F_1(\check{c}_1(\varepsilon)) - B_0\,\check{c}_1(\varepsilon).$$

Indeed, let us denote the vector-functions [95, 96]

$$v(t,\varepsilon,\mu) := z_0(t,c_0^*) + \mu\, u_1(t,\varepsilon) + \ldots + \mu^k\, u_k(t,\varepsilon) + \ldots,$$
$$g(\varepsilon,\mu) := \tau_0^* + \mu\,\xi_1(\varepsilon) + \mu^2\,\xi_2(\varepsilon) + \ldots + \mu^k\,\xi_k(\varepsilon) + \ldots,$$

herewith

$$F_1(c_1(\varepsilon),\xi_1(\varepsilon)) = P_{Q_d^*}\ell K[Z_1(z_0(s,c_0^*), u_1(s,\varepsilon),\varepsilon);\xi_1(\varepsilon)](\cdot)$$
$$= P_{Q_d^*}\,\ell K\big[Z_\mu'(v(t,\varepsilon,\mu),\varepsilon); g_\mu'(\varepsilon,\mu)\big](\cdot)\Big|_{\mu=0}$$

$$= P_{Q_d^*}\,\ell K[\mathcal{A}_0(s)u_1(s,\varepsilon);\xi_1(\varepsilon)](\cdot);$$

therefore,

$$B_0 := F'_{\check{c}_1(\varepsilon)}(\check{c}_1(\varepsilon)) \in \mathbb{R}^{r\times(r+1)}.$$

Thus, under the condition (13), we obtain at least one solution to the first approximation boundary-value problem

$$z_1(t,\varepsilon) := z_0(t,c_0^*) + u_1(t,\varepsilon), \quad \tau_1(\varepsilon) = \tau_0^* + \xi_1(\varepsilon), \quad \check{c}_1(\varepsilon) = -B_0^+\,\gamma_1(\varepsilon),$$
$$u_1(t,\varepsilon) = X_r(t)c_1(\varepsilon) + \varepsilon\,G\big[Z_1\big(z_0(s,c_0^*),u_1'(s,\varepsilon),\varepsilon\big);\xi_1(\varepsilon)\big](t).$$

The conditions of solvability of boundary-value problems for the approximations

$$F_j(\check{c}_j(\varepsilon)) := P_{Q_d^*}\,\ell K[Z_j(z_0(s,c_0^*),u_1(t,\varepsilon),\ldots,u_j(s,\varepsilon),\xi_j(\varepsilon),\varepsilon)](\cdot) = 0$$

are linear equations

$$F_j(\check{c}_j(\varepsilon)) = B_0\,\check{c}_j(\varepsilon) + \gamma_j(\varepsilon) = 0,$$

where

$$B_0 = F'(\check{c}_j(\varepsilon)), \quad \gamma_j(\varepsilon) := F(\check{c}_j(\varepsilon)) - B_0\,\check{c}_j(\varepsilon), \quad j = 1,2,\ldots,k.$$

In the case (13) the last equation is at least uniquely solvable. The sequence of approximations to the solution of the nonlinear periodic boundary-value problem (5) in the critical case is determined by the iterative scheme

$$\begin{gathered}
z_1(t,\varepsilon) = z_0(t,c_0^*) + u_1(t,\varepsilon), \quad \tau_1(\varepsilon) = \tau_0^* + \xi_1(\varepsilon), \quad \check{c}_1(\varepsilon) = -B_0^+\,\gamma_1(\varepsilon),\\
u_1(t,\varepsilon) = X_r(t)c_1(\varepsilon) + \varepsilon\,G\big[Z_1\big(z_0(s,c_0^*),u_1'(s,\varepsilon),\varepsilon\big);\xi_1(\varepsilon)\big](t),\\
z_2(t,\varepsilon) = z_0(t,c_0^*) + u_1(t,\varepsilon) + u_2(t,\varepsilon), \quad \tau_2(\varepsilon) = \tau_0^* + \xi_1(\varepsilon) + \xi_2(\varepsilon),\\
u_2(t,\varepsilon) = X_r(t)c_2(\varepsilon) + \varepsilon\,G[Z_2(z_0(s,c_0^*),u_1(s,\varepsilon),u_2(s,\varepsilon),\varepsilon);\xi_2(\varepsilon)](t),\\
z_{k+1}(t,\varepsilon) = z_0(t,c_0^*) + u_1(t,\varepsilon) + \ldots + u_{k+1}(t,\varepsilon),\\
\tau_{k+1}(\varepsilon) = \tau_0^* + \xi_1(\varepsilon) + \ldots + \xi_{k+1}(\varepsilon), \quad u_{k+1}(t,\varepsilon) = X_r(t)\,c_{k+1}(\varepsilon)\\
+\,\varepsilon\,G[Z_k(z_0(s,c_0^*),u_1(s,\varepsilon),\ldots,u_k(s,\varepsilon),\varepsilon);\xi_k(\varepsilon)](t), \quad k = 0,1,2,\ldots.
\end{gathered} \tag{14}$$

Theorem. *Suppose that there is the critical case of the generating boundary-value problem (6). In this case, the generating problem (6) has a family of solutions*

$$z_0(t,c_0) = X_r(t)\,c_0, \quad c_0 \in \mathbb{R}^r.$$

In the case of (13) in the small neighbourhood of the generating solution $z_0\big(t,c_0^*\big)$ and the constant τ_0^* the problem (5) with switchings has at least a unique solution. The sequence of approximations to the solution

$$z(\cdot,\varepsilon) \in \mathbb{C}\{[0,T]\setminus\{\tau(\varepsilon)\}_I\} \cap \mathbb{C}[0,T], \quad z(t,\cdot) \in \mathbb{C}[0,\varepsilon_0]$$

of the autonomous boundary-value problem (5) with switchings is determined by an iterative scheme (14). If there exist constants $0 < \gamma < 1$ *and* $0 < \delta < 1$ *such that inequalities*

$$\|u_1(t,\varepsilon)\| \le \gamma \|z_0(t,c_0^*)\|, \quad \|u_{k+1}(t,\varepsilon)\| \le \gamma \|u_k(t,\varepsilon)\|,$$
$$|\xi_1(\varepsilon)| \le \delta\, |\tau_0^*|, \quad |\xi_{k+1}(\varepsilon)| \le q\delta\, |xi_k(\varepsilon)|, \quad k = 1, 2, \ldots, \tag{15}$$

are hold, then the iterative scheme (14) converges to the solution of the autonomous boundary-value problem (5) with switchings.

The obtained convergence condition (15) of the iterative scheme (14) allows us to estimate the interval of values of the small parameter $\varepsilon \in [0, \varepsilon_0],\ 0 \le \varepsilon_* \le \varepsilon_0,$ for which there is a retention the convergence of the iterative scheme (14) and different from similar estimates [97, 98].

3. Finding approximations to the periodic solution of the equation with switchings modelling a nonisothermal chemical reaction. Let us apply our iterative scheme (14) to find approximations to the periodic solution of the equation modelling a nonisothermal chemical reaction [72, 99]. The specificity of such models is the fact that in the absence of switching, the equation modelling a nonisothermal chemical reaction usually has a single solution, which is the equilibrium position [72]. Moreover, the behaviour of solutions of the equation modelling a nonisothermal chemical reaction with switchings at nonfixed points in time well reflects the behaviour of the periodic solution of the equation modelling a nonisothermal chemical reaction studied in the articles [72, 99].

Example 3. Let us demonstrate the effectiveness of the theorem on the example of the problem of finding 2π-periodic solutions

$$z(\cdot,\varepsilon) \in \mathbb{C}^2\{[0, 2\pi] \setminus \{\tau(\varepsilon)\}_I\} \cap \mathbb{C}[0, 2\pi], \quad z(t,\cdot) \in \mathbb{C}[0, \varepsilon_0]$$

of the nonlinear equation with switchings

$$z'(t,\varepsilon) = \omega(\varepsilon)\, A\, z(t,\varepsilon) + \varepsilon Z(z(t,\varepsilon),\varepsilon),$$
$$z(0,\varepsilon) - z(2\pi,\varepsilon) = 0, \quad z(0,\varepsilon) - z(\tau(\varepsilon),\varepsilon) = 0, \tag{16}$$

continuous at $t = \tau(\varepsilon)$. At the point $t = \tau(\varepsilon)$:

$$0 < \tau(\varepsilon) < 2\pi, \quad \tau(0) := \tau_0$$

the solution of the boundary-value problem (16) might have a limited discontinuity of the first derivative. The solution of the boundary-value problem (16) with switchings is found in a small neighbourhood of the solution

$$z_0(t) \in \mathbb{C}\{[0, 2\pi] \setminus \{\tau_0\}_I\} \cap \mathbb{C}[0, 2\pi]$$

of the generating boundary-value problem. Here,

$$z(t,\varepsilon) := \begin{cases} x(t,\varepsilon), & t \in [0, \tau(\varepsilon)[, \\ y(t,\varepsilon), & t \in [\tau(\varepsilon), 2\pi], \end{cases} \qquad w(\varepsilon) := \begin{cases} 1-\varepsilon, & t \in [0, \tau(\varepsilon)[, \\ 1+\varepsilon, & t \in [\tau(\varepsilon), 2\pi], \end{cases}$$

moreover,

$$Z(z(t,\varepsilon),\varepsilon) := (1 + x(t,\varepsilon))\, e^{-\frac{\varepsilon}{1+y(t,\varepsilon)}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad t \in [0, \tau(\varepsilon)],$$

$$Z(z(t,\varepsilon),\varepsilon) := (1+y(t,\varepsilon))\, e^{-\frac{\varepsilon}{1+x(t,\varepsilon)}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad t \in [\tau(\varepsilon), 2\pi].$$

There is the critical case for the generating boundary-value problem. Thus, the generating problem has a two-parameter family of solutions

$$z_0(t, c_0) = X_r(t)\, c_0, \quad X_r(t) := \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \quad c_0 \in \mathbb{R}^2.$$

Let us put

$$c_0 := \begin{pmatrix} c_{0a} \\ c_{0b} \end{pmatrix}, \quad \tau_0^* := \frac{\pi}{2}.$$

To find the amplitude c_0 of the generating solution, we obtain the equation

$$F_0(c_0, \tau_0^*) = \pi \begin{pmatrix} c_{0b} \\ c_{0a} \end{pmatrix} = 0,$$

herewith

$$B_0 = \begin{pmatrix} 0 & \pi & 0 \\ -\pi & 0 & 0 \end{pmatrix}, \quad P_{B_0^*} = 0.$$

Hence,

$$c_0^* = 0, \quad \tau_0^* = \frac{\pi}{2}.$$

By using the iterative scheme (14) at the first step, we obtain

$$z_1(t,\varepsilon) = z_0(t, c_0^*) + u_1(t,\varepsilon), \quad t \in [0, 2\pi],$$

$$u_1(t,\varepsilon) = X_r(t)\, c_1(\varepsilon) + \varepsilon\, G[Z_1(z_0(s, c_0^*), u_1(s,\varepsilon), \varepsilon); \xi_1(\varepsilon)](t),$$

while

$$u_1(t,\varepsilon) = \frac{\varepsilon}{\pi} \begin{pmatrix} \pi - 4\sin t \\ -\pi - 4\cos t \end{pmatrix}, \quad \xi_1(\varepsilon) = \varepsilon\, \tau_0^*, \quad t \in [0, 2\pi].$$

In the second step, we get

$$z_2(t,\varepsilon) := z_0(t, c_0^*) + u_1(t,\varepsilon) + u_2(t,\varepsilon), \quad \tau_2(\varepsilon) = \tau_0^* + \xi_1(\varepsilon) + \xi_2(\varepsilon), \quad \xi_2(\varepsilon) = \varepsilon^2\, \tau_0^*,$$

$$u_2(t,\varepsilon) = X_r(t) c_2(\varepsilon) + \varepsilon\, G[Z_2(z_0(s, c_0^*), u_1(s,\varepsilon), u_2(s,\varepsilon), \varepsilon); \xi_2(\varepsilon)](t),$$

herewith

$$u_2(t,\varepsilon) = \frac{\varepsilon^2}{\pi^2} \begin{pmatrix} \pi^2 + (4 - 2\pi^2 + \pi(2+6t))\cos t \\ -\pi^2 + (4\cos t + \pi^2 - 2\pi(1+t))\cos t \end{pmatrix}$$

$$+ \frac{\varepsilon^2}{\pi^2} \begin{pmatrix} (4 + \pi^2 - 2\pi\,(2+t))\sin t \\ 2\,(\pi^2 - 2 - 3\,\pi t)\sin t \end{pmatrix}, \quad t \in [0, \tau_2(\varepsilon)],$$

$$u_2(t,\varepsilon) = \frac{\varepsilon^2}{\pi^2}\begin{pmatrix} -\pi^2 + 2\,(2+\pi+2\pi^2 - \pi\,t)\cos t \\ \pi^2 + (4+3\pi^2 - 2\pi\,(1+t))\cos t \end{pmatrix}$$

$$+\,\frac{\varepsilon^2}{\pi^2}\begin{pmatrix} (4+3\,\pi^2 - 2\pi\,(2+t))\sin t \\ (-4-4\pi^+2\pi\,t)\sin t \end{pmatrix}, \quad t \in [\tau_2(\varepsilon), 2\pi].$$

Note that for any value of the small parameter

$$\varepsilon \in [0, \varepsilon_0], \quad \varepsilon_0 \approx 0.657\,585,$$

the inequalities (15) are hold. This inequalities indicate the practical convergence of the obtained approximations to the solution of the periodic problem for the nonlinear equation (16), where $\gamma := 0.997 < 1$. Note also that the approximations satisfy the boundary condition (16).

The accuracy of the found approximations to the solution of a periodic problem for a nonlinear equation (16) is determined by the following inequalities:

$$\Delta_k(\varepsilon) := \left\| z_k'(t,\varepsilon) - \omega(\varepsilon)\,A\,z_k(t,\varepsilon) - \varepsilon Z(z_k(t,\varepsilon),\varepsilon)\right\|_{\mathbb{C}[0,2\pi]}, \quad k = 0,1,2.$$

In particular,

$$\Delta_0(0.1) \approx 0.127\,963, \quad \Delta_1(0.1) \approx 0.0293\,051, \quad \Delta_2(0.1) \approx 0.00493\,865,$$

$$\Delta_0(0.01) \approx 0.0140\,014, \quad \Delta_1(0.01) \approx 0.000\,252\,071, \quad \Delta_2(0.01) \approx 4.77\,925 \times 10^{-6}.$$

The example of a periodic problem for a nonlinear equation (16) demonstrates that, due to switching, the autonomous boundary-value problem becomes solvable on a fixed length interval, in contrast to similar autonomous boundary-value problems without switching, for which, generally speaking, the right end $T(\varepsilon)$ of the interval $[0, T(\varepsilon)]$, where the desired solution of the nonlinear boundary-value problem for the system (5) is defined, is unknown and to be determined in the process of solution construction [88, 89, 96, 97]. To construct solutions to the nonlinear boundary-value problem for the system (5) the least-squares method [100, 101] is also applicable, in particular, for systems with switchings [77].

Proposed in the paper conditions of solvability and scheme of constructing solutions of nonlinear boundary-value problems with switching at nonfixed points in time with a Noether linear part can be transferred to boundary-value problems with switching at nonfixed points in time with a normally solvable linear part [102], as well as to hybrid boundary-value problems [103].

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