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# ADOMIAN DECOMPOSITION METHOD IN THE THEORY OF NONLINEAR BOUNDARY-VALUE PROBLEMS UNSOLVED WITH RESPECT TO THE DERIVATIVE

# МЕТОД ДЕКОМПОЗИЦІЇ АДОМЯНА У ТЕОРІЇ НЕЛІНІЙНИХ КРАЙОВИХ ЗАДАЧ, НЕ РОЗВ'ЯЗАНИХ ЩОДО ПОХІДНОЇ

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We find constructive necessary and sufficient conditions for solvability and a scheme for construction of solutions of a nonlinear boundary-value problem unsolved with respect to the derivative. On the basis of the Adomian decomposition method, convergent iterative schemes for finding approximations of solutions of a nonlinear boundary-value problem unsolved with respect to the derivative are constructed. As examples of the application of the constructed iterative scheme, we find approximations of the solutions of a periodic boundary-value problem for Rayleigh-type equations unsolved with respect to the derivative including the case of a periodic problem for the equation that determines the motion of a satellite in an elliptical orbit.

Знайдено конструктивні необхідні й достатні умови розв'язності та схему побудови розв'язків нелінійної крайової задачі, не розв'язаної щодо похідної. На основі методу декомпозиції Адомяна побудовано збіжні ітераційні схеми для знаходження наближень до розв'язків нелінійної крайової задачі, не розв'язаної щодо похідної. Як приклади застосування побудованої ітераційної схеми одержано наближення до розв'язків періодичних крайових задач для рівнянь типу Релея, не розв'язаного щодо похідної, зокрема у випадку періодичної задачі для рівняння, яке визначає рух супутника на еліптичний орбіті.

**1. Statement of the problem.** We study the problem of constructing solutions

$$z(t,\varepsilon)\colon z(\cdot,\varepsilon)\in\mathbb{C}^1[a,b],\quad z(t,\cdot)\in\mathbb{C}[0,\varepsilon_0]$$

of the boundary-value problem [1 – 3]

$$z'(t,\varepsilon)=A(t)\,z(t,\varepsilon)+f(t)+\varepsilon\,Z(z(t,\varepsilon),z'(t,\varepsilon),t,\varepsilon), \tag{1}$$

$$\ell z(\cdot,\varepsilon)=\alpha+\varepsilon\,J(z(\cdot,\varepsilon),z'(\cdot,\varepsilon),\varepsilon) \tag{2}$$

in a small neighborhood of the solution of the generating Noetherian $(m\neq n)$ boundary-value problem

$$z_0'(t)=A(t)z_0(t)+f(t),\qquad \ell z_0(\cdot)=\alpha,\quad \alpha\in\mathbb{R}^m. \tag{3}$$

Here, $A(t)$ is an $(n\times n)$-matrix, $f(t)$ is an $n$-dimensional column vector whose elements are real functions continuous on the segment $[a,b]$ and $\ell z(\cdot,\varepsilon)$ is linear and $J(z(\cdot,\varepsilon),z'(\cdot,\varepsilon),\varepsilon)$ is nonlinear vector functionals

$$\ell z(\cdot,\varepsilon),\qquad J(z(\cdot,\varepsilon),z'(\cdot,\varepsilon),\varepsilon)\colon\mathbb{C}[a,b]\to\mathbb{R}^m,$$

where the second functional is continuously differentiable in the unknowns $z,\ z'$ and in a small parameter $\varepsilon$ in a small neighbourhood of the solution of the generating problem and its derivative and on the interval $[0,\varepsilon_0]$. We assume that the nonlinearity $Z(z,z',t,\varepsilon)$ of the boundary-value problem (1), (2) is analytical in the unknown $z$ and its derivative $z'$ in a small neighborhood of the generating solution and its derivative, and in a small parameter $\varepsilon$ in a small positive neighborhood of zero. In addition, we assume that the nonlinear vector function $Z(z,z',t,\varepsilon)$ is continuous in the independent variable $t$ on the interval $[a,b]$.

The relevance of studying the nonautonomous boundary-value problem (1), (2), unsolved with respect to the derivative, is related to the fact that the study of the traditional problem [1], which is solved with respect to the derivative, is sometimes complicated, for example, in the case of nonlinearities that are not integrable in elementary functions. An example of a similar situation is given in the articles [2, 4 – 6]. An example of a similar situation can also be an autonomous boundary-value problem unsolved with respect to the derivative, in particular, the periodic problem for the Lotka equation [3, 7].

In the article [5], the constructive necessary and sufficient conditions for solvability and the scheme for constructing solutions to the nonlinear boundary-value problem (1), (2) are found and, based on the method of simple iterations, a convergent iteration scheme to find approximations to the solutions of this problem is constructed. At the same time, when constructing solutions to the nonlinear boundary-value problem (1), (2) with the iterative scheme [5], the problem of the impossibility of finding solutions in elementary functions once again arises, which, in turn, leads to large errors in the solutions of nonlinear boundary-value problems.

In addition, the construction of solutions of nonlinear boundary-value problems by using the method of simple iterations [1] is significantly complicated by the calculation of derivative nonlinearities. In the article [8], the acceleration of the convergence of iterative schemes is achieved by calculating the derivatives of nonlinearities at each step. Given the above, simplification of the calculation of derivative nonlinearities and the possibility of finding solutions of nonlinear boundary-value problems, in particular, periodic boundary-value problems, in elementary functions can be achieved by using the Adomian decomposition method [9]. In particular, the use of the Adomian decomposition method significantly simplifies the proof of convergence of iterative schemes for constructing solutions of nonlinear boundary-value problems. An example of this simplification will be given below.

**2. Conditions of solvability.** We study the critical case $(P_{Q^*} \neq 0)$ and assume that the condition

$$P_{Q_d^*}\big\{\alpha - \ell K[f(s)](\cdot)\big\} = 0 \tag{4}$$

is fulfilled. In this case, the generating problem (3) has a family of solutions

$$z_0(t, c_r) = X_r(t)c_r + G[f(s); \alpha](t), \quad r := n - n_1, \quad c_r \in \mathbb{R}^r.$$

Here, $X(t)$ is a normal $(X(a) = I_n)$ fundamental matrix of the homogeneous part of the system (3), $Q = \ell X(\cdot)$ is an $(m \times n)$-dimensional matrix,

$$\operatorname{rank} Q = n_1, \qquad X_r(t) = X(t)P_{Q_r},$$

$P_{Q_r}$ is an $(n \times r)$-dimensional matrix formed from $r$ linearly independent columns of an $(n \times n)$-dimensional orthoprojector matrix

$$P_Q : \mathbb{R}^n \to \mathbb{N}(Q),$$

$P_{Q_d^*}$ is an $(d \times m)$-dimensional matrix formed from $(d := m - n_1)$ linearly independent rows of an $(m \times m)$-dimensional orthoprojector matrix

$$P_{Q^*} : \mathbb{R}^m \to \mathbb{N}(Q^*).$$

In addition,

$$G[f(s); \alpha](t) = K[f(s)](t) + X(t)Q^+\big\{\alpha - \ell K[f(s)](\cdot)\big\}$$

is a generalized Green's operator of the boundary-value problem (3),

$$K[f(s)](t) = X(t)\int_a^t X^{-1}(s)f(s)\,ds,$$

$Q^+$ is a Moore – Penrose pseudo-inverse matrix [1]. The solution of the problem (1), (2) in the critical case is found in the form

$$z(t, \varepsilon) := z_0(t, c_r) + u_1(t, \varepsilon) + \ldots + u_k(t, \varepsilon) + \ldots .$$

The nonlinear vector-function $Z(z, z', t, \varepsilon)$ is analytical in the unknown $z$ and its derivative $z'$ in the neighborhood of the solution of the generating problem (3), so in this neighborhood the following expansion converges [9, p. 502]:

$$\begin{aligned} Z(z(t,\varepsilon), z'(t,\varepsilon), t, \varepsilon) = A_0(z_0(t, c_r)) &+ A_1(z_0(t, c_r), u_1(t, \varepsilon), \varepsilon) \\ &+ A_2(z_0(t, c_r), u_1(t, \varepsilon), u_2(t, \varepsilon), \varepsilon) + \ldots \\ &+ A_n(z_0(t, c_r), u_1(t, \varepsilon), \ldots, u_n(t, \varepsilon), \varepsilon) + \ldots . \end{aligned}$$

The nonlinear bounded vector functional $J(z(\cdot,\varepsilon),\varepsilon)$ is analytical in the unknown $z$ in a small neighbourhood of the solution of the generating problem (3) and its derivative, so the following expansion takes place in this neighbourhood:

$$\begin{aligned} J(z(\cdot,\varepsilon), z'(\cdot,\varepsilon),\varepsilon) = J_0(z_0(\cdot,c_r)) + J_1(z_0(\cdot,c_r),u_1(\cdot,\varepsilon)) \\ + J_2(z_0(\cdot,c_r),u_1(\cdot,\varepsilon),u_2(\cdot,\varepsilon)) + \ldots \\ + J_k(z_0(\cdot,c_r),u_1(\cdot,\varepsilon),u_2(\cdot,\varepsilon),\ldots,u_k(\cdot,\varepsilon)) + \ldots . \end{aligned}$$

The expansions of many nonlinear functions and formulas for their calculation are presented in the articles [9 – 11]. Necessary conditions for the existence of a solution of the problem (1), (2) in the critical case are defined by the following lemma. The proof of the lemma is similar to [1, 3, 5, 12].

**Lemma.** *Suppose that for the boundary-value problem (1), (2) there is a critical* $(P_{Q^*} \neq 0)$ *case and the condition (4) is satisfied for the solvability of the generating problem (3). Let us also assume that the problem (1), (2) has a solution that for* $\varepsilon = 0$ *transforms into the generating* $z_0\big(t,c_r^*\big)$. *Then the vector* $c_r^* \in \mathbb{R}^r$ *satisfies the equation*

$$F_0(c_r) := P_{Q_d^*}\big\{J_0(z_0(\cdot,c_r)) - \ell K[A_0(z_0(s,c_r))](\cdot)\big\} = 0. \tag{5}$$

By analogy with the weakly nonlinear boundary-value problems in the critical case [1], the equation (5) will be called the equation for the generating constants of the boundary-value problem (1), (2) unsolved with respect to the derivative. Let us assume that the equation (5) has real roots and does not transform into an identity [13, 14]. Fixing one of the solutions $c_r^* \in \mathbb{R}^r$ of the equation (5), we come to the problem of finding analytical solutions of the boundary-value problem (1), (2) in a small neighborhood of the solution

$$z_0\big(t,c_r^*\big) = X_r(t)c_r^* + G[f(s);\alpha](t), \quad c_r^* \in \mathbb{R}^r,$$

of the generating problem (3). The first approximation to the solution of the nonlinear boundary-value problem (1), (2) unsolved with respect to the derivative in the critical case

$$z_1(t,\varepsilon) := z_0\big(t,c_r^*\big) + u_1(t,\varepsilon), \quad u_1(t,\varepsilon) = X_r(t)c_1 + \varepsilon\, G\big[A_0\big(z_0(s,c_r^*)\big); J_0\big(z_0\big(\cdot,c_r^*\big)\big)\big](t)$$

determines the solution of the nonlinear boundary-value problem of the first approximation

$$\frac{du_1(t,\varepsilon)}{dt} = A(t)\,u_1(t,\varepsilon) + \varepsilon\, A_0\big(z_0\big(t,c_r^*\big)\big), \quad \ell u_1(\cdot,\varepsilon) = \varepsilon\, J_0\big(z_0\big(\cdot,c_r^*\big)\big).$$

The solvability of the first approximation boundary-value problem is guaranteed by the choice of the solution $c_r^* \in \mathbb{R}^r$ of the equation (5). The second approximation to the solution of the nonlinear boundary-value problem (1), (2) unsolved with respect to the derivative in the critical case

$$z_2(t,\varepsilon) := z_0\big(t,c_r^*\big) + u_1(t,\varepsilon) + u_2(t,\varepsilon)$$

determines the solution of the nonlinear periodic boundary-value problem of the second approximation

$$\frac{du_2(t,\varepsilon)}{dt} = A(t)\,u_2(t,\varepsilon) + \varepsilon\, A_1\big(z_0\big(t,c_r^*\big),u_1(t,\varepsilon)\big), \qquad \ell u_2(\cdot,\varepsilon) = \varepsilon\, J_1\big(z_0\big(\cdot,c_r^*\big),u_1(\cdot,\varepsilon)\big),$$

where

$$u_2(t,\varepsilon) = X_r(t)c_2 + \varepsilon\, G\big[A_1\big(z_0\big(s,c_r^*\big),u_1(s,\varepsilon)\big);J_1\big(z_0\big(\cdot,c_r^*\big),u_1(\cdot,\varepsilon)\big)\big](t), \quad c_2\in\mathbb{R}^r.$$

In a small neighborhood of the generating solution $z_0\big(t,c_r^*\big)$, the following expansion is valid:

$$\begin{aligned} &Z\big(z_0\big(t,c_r^*\big)+x(t,\varepsilon),z_0'\big(t,c_r^*\big)+x'(t,\varepsilon),t,\varepsilon\big)\\ &\quad = Z\big(z_0\big(t,c_r^*\big),z_0'\big(t,c_r^*\big),t,0\big)+\mathcal{A}_1(t)x(t,\varepsilon)\\ &\qquad + \mathcal{A}_2(t)x'(t,\varepsilon)+\varepsilon\,\mathcal{A}_3(t)+R\big(z_0\big(t,c_r^*\big)+x(t,\varepsilon),z_0'\big(t,c_r^*\big)+x'(t,\varepsilon),t,\varepsilon\big),\end{aligned}$$

where

$$\mathcal{A}_1(t) := \frac{\partial Z(z,z',t,\varepsilon)}{\partial z}\bigg|_{\substack{z=z_0(t,c_r^*),\\ z'=z_0'(t,c_r^*),\\ \varepsilon=0,}} \qquad \mathcal{A}_2(t) := \frac{\partial Z(z,z',t,\varepsilon)}{\partial z'}\bigg|_{\substack{z=z_0(t,c_r^*),\\ z'=z_0'(t,c_r^*),\\ \varepsilon=0,}}$$

$$\mathcal{A}_3(t) := \frac{\partial Z(z,z',t,\varepsilon)}{\partial \varepsilon}\bigg|_{\substack{z=z_0(t,c_r^*),\\ z'=z_0'(t,c_r^*),\\ \varepsilon=0.}}$$

Under the assumption of sufficient smallness of $x,\ x'$ and $\varepsilon$ the residual

$$R_1\big(z_0\big(t,c_r^*\big)+x(t,\varepsilon),z_0'\big(t,c_r^*\big)+x'(t,\varepsilon),t,\varepsilon\big)$$

of the higher order in the neighborhood of the points $x=0,\ x'=0$ and $\varepsilon=0$ for the first four terms of the expansion for

$$R(z,z',t,\varepsilon)\Big|_{\substack{z=z_0(t,c_r^*),\\ z'=z_0'(t,c_r^*),\\ \varepsilon=0,}} \equiv 0, \qquad \frac{\partial R(z,z',t,\varepsilon)}{\partial z}\bigg|_{\substack{z=z_0(t,c_r^*),\\ z'=z_0'(t,c_r^*),\\ \varepsilon=0}} \equiv 0,$$

$$\frac{\partial R(z,z',t,\varepsilon)}{\partial z'}\bigg|_{\substack{z=z_0(t,c_r^*),\\ z'=z_0'(t,c_r^*),\\ \varepsilon=0}} \equiv 0, \qquad \frac{\partial R(z,z',t,\varepsilon)}{\partial \varepsilon}\bigg|_{\substack{z=z_0(t,c_r^*),\\ z'=z_0'(t,c_r^*),\\ \varepsilon=0}} \equiv 0.$$

In a small neighborhood of the generating solution $z_0\big(t,c_r^*\big)$, the following expansion is valid:

$$\begin{aligned} &J\big(z_0\big(\cdot,c_r^*\big)+x(\cdot,\varepsilon),z_0'\big(\cdot,c_r^*\big)+x'(\cdot,\varepsilon),\varepsilon\big)\\ &\quad = J\big(z_0\big(\cdot,c_r^*\big),z_0'\big(\cdot,c_r^*\big),0\big)+\ell_1\,x(\cdot,\varepsilon)\\ &\qquad + \ell_2\,x'(\cdot,\varepsilon)+\varepsilon\,\ell_3\big(z_0\big(\cdot,c_r^*\big)\big)+\mathcal{J}(z(\cdot,\varepsilon)+x(\cdot,\varepsilon),z'(\cdot,\varepsilon)+x'(\cdot,\varepsilon),\varepsilon).\end{aligned}$$

Here,

$$\ell_1 x(\cdot,\varepsilon) := \frac{\partial J(z(\cdot,\varepsilon),z'(\cdot,\varepsilon),\varepsilon)}{\partial z}\bigg|_{\substack{z(\cdot,\varepsilon)=z_0(t,c_r^*),\\ z'(\cdot,\varepsilon)=z_0'(t,c_r^*),\\ \varepsilon=0,}}$$

$$\ell_2 x(\cdot,\varepsilon) := \frac{\partial J(z(\cdot,\varepsilon),z'(\cdot,\varepsilon),\varepsilon)}{\partial z'}\bigg|_{\substack{z(\cdot,\varepsilon)=z_0(t,c_r^*),\\ z'(\cdot,\varepsilon)=z_0'(t,c_r^*),\\ \varepsilon=0,}}$$

$$\ell_3\big(z_0(\cdot, c_r^*)\big) := \frac{\partial J(z(\cdot,\varepsilon), z'(\cdot,\varepsilon), \varepsilon)}{\partial \varepsilon}\bigg|_{\substack{z(\cdot,\varepsilon)=z_0(t,c_r^*),\\ z'(\cdot,\varepsilon)=z_0'(t,c_r^*),\\ \varepsilon=0}}$$

are Frechet derivatives of the vector functional

$$J(z(\cdot,\varepsilon)+x(\cdot,\varepsilon), z'(\cdot,\varepsilon)+x'(\cdot,\varepsilon), \varepsilon).$$

Under the assumption of sufficient smallness of $x,\ x'$ and $\varepsilon$ the residual

$$\mathcal{J}(z(\cdot,\varepsilon)+x(\cdot,\varepsilon), z'(\cdot,\varepsilon)+x'(\cdot,\varepsilon), \varepsilon)$$

of the higher order in the neighborhood of the points $x=0,\ x'=0$ and $\varepsilon=0$ for the first four terms of the expansion for

$$\mathcal{J}(z(\cdot,\varepsilon)+x(\cdot,\varepsilon), z'(\cdot,\varepsilon)+x'(\cdot,\varepsilon), \varepsilon)\Big|_{\substack{z=z_0(t,c_r^*),\\ z'=z_0'(t,c_r^*),\\ \varepsilon=0}} \equiv 0,$$

$$\frac{\partial \mathcal{J}(z(\cdot,\varepsilon)+x(\cdot,\varepsilon), z'(\cdot,\varepsilon)+x'(\cdot,\varepsilon), \varepsilon)}{\partial z}\bigg|_{\substack{z=z_0(t,c_r^*),\\ z'=z_0'(t,c_r^*),\\ \varepsilon=0}} \equiv 0,$$

$$\frac{\partial \mathcal{J}(z(\cdot,\varepsilon)+x(\cdot,\varepsilon), z'(\cdot,\varepsilon)+x'(\cdot,\varepsilon), \varepsilon)}{\partial z'}\bigg|_{\substack{z=z_0(t,c_r^*),\\ z'=z_0'(t,c_r^*),\\ \varepsilon=0}} \equiv 0,$$

$$\frac{\partial \mathcal{J}(z(\cdot,\varepsilon)+x(\cdot,\varepsilon), z'(\cdot,\varepsilon)+x'(\cdot,\varepsilon), \varepsilon)}{\partial \varepsilon}\bigg|_{\substack{z=z_0(t,c_r^*),\\ z'=z_0'(t,c_r^*),\\ \varepsilon=0}} \equiv 0.$$

Define the $(d \times r)$-dimensional matrix

$$B_0 := P_{Q_d^*}\big\{\ell_1 X_r(\cdot) + \ell_2 X_r'(\cdot) - \ell K\big[\mathcal{A}_1(s)X_r(s) + \mathcal{A}_2(s)X_r'(s)\big](\cdot)\big\}.$$

The traditional condition for the solvability of the problem (1), (2) unsolved with respect to the derivative, in the small neighborhood of the solution of the generating problem (3) is the requirement of simplicity of roots [1, 5, 12]

$$P_{B_0^*} P_{Q_d^*} = 0 \tag{6}$$

of the equations (3) of the generating amplitudes of the problem (1), (2). Here, $P_{B_0^*}$ is a $(d \times d)$-dimensional orthoprojector matrix

$$\mathbb{R}^d \to \mathbb{N}\big(B_0^*\big).$$

The condition for solvability of the boundary-value problem of the second approximation

$$F_1(c_1) := P_{Q_d^*}\big\{J_1\big(z_0(\cdot, c_r^*), u_1(\cdot,\varepsilon)\big) - \ell K\big[A_1\big(z_0(s, c_r^*), u_1(s,\varepsilon)\big)\big](\cdot)\big\} = 0$$

is a linear equation

$$F_1(c_1) = B_0\, c_1 + d_1 = 0,$$

which is solvable under the condition (6). Indeed, let us denote the vector function [15]

$$v(t,\mu) := z_0\big(t, c_r^*\big) + \mu\, u_1(t,\varepsilon) + \ldots + \mu^k\, u_k(t,\varepsilon) + \ldots,$$

while

$$\begin{aligned} F_1(c_1) &:= P_{Q_d^*}\big\{J_1\big(z_0\big(\cdot, c_r^*\big), u_1(\cdot,\varepsilon)\big) - \ell K\big[A_1\big(z_0\big(s, c_r^*\big), u_1(s,\varepsilon)\big)\big](\cdot)\big\} \\ &= P_{Q_d^*}\big\{J'_\mu(v(\cdot,\varepsilon), v'(\cdot,\varepsilon),\varepsilon) - \ell K\big[Z'_\mu(v(s,\mu), v'(s,\mu), s,\varepsilon)\big](\cdot)\,\big\}\Big|_{\mu=0} \\ &= P_{Q_d^*}\big\{\ell_1\, u_1(\cdot,\varepsilon) + \ell_2\, u_1'(\cdot,\varepsilon) - \ell K\big[\mathcal{A}_1(s) u_1(s,\varepsilon) + \mathcal{A}_2(s) u_1'(s,\varepsilon)\big](\cdot)\big\}. \end{aligned}$$

Hence,

$$B_0 = F_1'(c_1).$$

Thus, under the condition (6), we obtain at least one solution of the first approximation boundary-value problem

$$u_1(t,\varepsilon) = X_r(t)\, c_1 + \varepsilon\, G\big[A_0\big(z_0\big(s, c_r^*\big)\big); J_0\big(z_0\big(\cdot, c_r^*\big)\big)\big](t), \quad c_1 = -B_0^+\, d_1;$$

here

$$d_1 := F_1(c_1) - B_0\, c_1.$$

Conditions of solvability

$$\begin{aligned} F_j(c_j) := P_{Q_d^*}\Big\{ & J_j(z_0(\cdot, c_r), u_1(\cdot,\varepsilon), u_2(\cdot,\varepsilon), \ldots, u_j(\cdot,\varepsilon)) \\ & - \ell K\big[A_j\big(z_0\big(s, c_r^*\big), u_1(s, c_1), \ldots, u_j(s, c_j)\big)\big](\cdot)\Big\} = 0, \quad j = 1, 2, \ldots, k, \end{aligned}$$

for the boundary-value problems of the approximations

$$\frac{du_j(t,\varepsilon)}{dt} = A(t)\, u_j(t,\varepsilon) + \varepsilon\, A_j\big(z_0\big(t, c_r^*\big), u_1(t,\varepsilon), \ldots, u_j(t,\varepsilon)\big),$$

$$\ell u_j(\cdot,\varepsilon) = \varepsilon\, J_j(z_0(\cdot, c_r), u_1(\cdot,\varepsilon), u_2(\cdot,\varepsilon), \ldots, u_j(\cdot,\varepsilon))$$

are given by the linear equations

$$F_j(c_j) = B_0\, c_j + d_j = 0; \tag{7}$$

here

$$B_0 = F'(c_j) \in \mathbb{R}^{d\times r}, \qquad d_j := F(c_j) - B_0\, c_j, \quad j = 1, 2, \ldots, k.$$

In the case of (6) equation (7) is solvable. In the case of (6) the sequence of approximations to the solution of the nonlinear boundary-value problem (1), (2) unsolved with respect to the derivative in a small neighborhood of the solution of the generating problem (3) is determined by the iterative scheme

$$z_1(t,\varepsilon) := z_0\big(t, c_r^*\big) + u_1(t,\varepsilon), \qquad c_1 = -B_0^+\, d_1,$$

$$u_1(t,\varepsilon) = X_r(t)\, c_1 + \varepsilon\, G\big[A_0\big(z_0\big(s, c_r^*\big)\big); J_0\big(z_0\big(\cdot, c_r^*\big)\big)\big](t), \ldots,$$

$$z_{k+1}(t,\varepsilon) := z_0\big(t, c_r^*\big) + u_1(t,\varepsilon) + \ldots + u_{k+1}(t,\varepsilon), \tag{8}$$

$$u_{k+1}(t,\varepsilon) = X_r(t)\, c_{k+1} + \varepsilon\, G\Big[A_k\big(z_0\big(s, c_r^*\big), u_1(s,\varepsilon), \ldots, u_k(s,\varepsilon)\big),$$

$$J_k\big(z_0\big(\cdot, c_r^*\big), u_1(\cdot,\varepsilon), u_2(\cdot,\varepsilon), \ldots, u_k(\cdot,\varepsilon)\big)\Big](t),$$

$$c_k = -B_0^+\, d_k, \quad k = 0, 1, 2, \ldots .$$

The interval of values of the small parameter $\varepsilon \in [0, \varepsilon_0],\ 0 \le \varepsilon_* \le \varepsilon_0$, for which the convergence of the iterative scheme (8) to the solution of the nonlinear boundary-value problem (1), (2) unsolved with respect to the derivative, can be evaluated similarly to [1, 11, 15 – 17]. Thus, the following theorem is proved.

**Theorem.** *Suppose that for the boundary-value problem (1), (2) unsolved with respect to the derivative there exists a critical* $(P_{Q^*} \neq 0)$ *case and the condition (4) of solvability of the generating problem (3) is satisfied. Let us also assume that equation (5) does not transform into an identity and has real solutions. Then for each root* $c_r^* \in \mathbb{R}^r$ *of the equation (5) under the condition (6), the boundary-value problem (1), (2) unsolved with respect to the derivative has at least one analytical solution, which, for* $\varepsilon = 0$, *transforms into a generating solution*

$$z_0\big(t, c_r^*\big) = X_r(t) c_r^* + G[f(s); \alpha](t).$$

*This solution can be determined by using the iterative process (8). If there exists a constant* $0 < \gamma < 1$ *for* $\varepsilon \in \big[0, \varepsilon^*\big]$ *for which the inequality*

$$\|u_1(t,\varepsilon)\| \le \gamma \big\|z_0\big(t, c_r^*\big)\big\|, \quad \|u_{k+1}(t,\varepsilon)\| \le \gamma \|u_k(t,\varepsilon)\|, \quad k = 1, 2, \ldots, \tag{9}$$

*holds, then the iterative scheme (8) converges to the solution of the nonlinear boundary-value problem (1), (2) unsolved with respect to the derivative.*

In the case of $m = n$, for example, for periodic boundary-value problems, the matrix $B_0$ becomes square, and condition (6) transforms into the well-known [1, 16] requirement of nondegeneracy of the matrix $B_0$. Under the condition (6) we say that for the boundary-value problem (1), (2) unsolved with respect to the derivative, there is a critical case of the first order. In the case of $P_{B_0^*} P_{Q_d^*} \neq 0$ for the boundary-value problem (1), (2) there is a critical case of the second or higher order [1, 18]. The proved theorem generalizes the results of [1, 2] to the case of a nonlinear boundary-value problem unsolved with respect to the derivative.

**3. A periodic problem for a Rayleigh-type equation unsolved with respect to the derivative.** We will demonstrate a scheme for constructing solutions of the boundary-value problem (1), (2) unsolved with respect to the derivative on the example of a $T$-periodic problem for the Rayleigh-type equation unsolved with respect to the highest derivative [19, p. 177]

$$y'' = f(t) + \varepsilon\, Y\big(y, y', y'', t, \varepsilon\big). \tag{10}$$

Here, $Y(y, y', y'', t, \varepsilon)$ is a nonlinear scalar function analytic in the unknown $y$ and its derivatives $y'$ and $y''$ in a small neighborhood of the solution of the generating problem, continuous in $t$ on the interval $[0, T]$ and continuous in a small parameter $\varepsilon$ on the interval $[0, \varepsilon_0]$. The generating $T$-periodic problem

$$y_0'' = f(t), \qquad y(0) - y(T) = 0, \qquad y'(0) - y'(T) = 0$$

is critical

$$X(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & -2\pi \\ 0 & 0 \end{pmatrix}, \qquad X_r(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Let us assume that the generating $T$-periodic problem is solvable. For this, the following equality is fulfilled:

$$\int_0^T f(t)\,dt = 0.$$

If this requirement is fulfilled, then the solution of the generating problem has the form

$$y_0(t, c_0) = c_0 + g[f(s)](t), \quad c_0 \in \mathbb{R}^1,$$

where

$$g[f(s)](t) := k[f(s)](t) - \frac{t}{T}\int_0^T (T-s)f(s)\,ds$$

is the Green's operator of the generating $T$-periodic problem and

$$k[f(s)](t) := \int_0^t (t-s)f(s)\,ds$$

is the Green's operator of the Cauchy problem. We will search the periodic solutions of the Rayleigh type equation (10) in the neighborhood of the solution $y_0(t, c_0)$ of the linear part of this equation. Let us assume that the equation for the generating amplitudes in the case of a $T$-periodic problem for a Rayleigh-type equation

$$F_0(c_0) := \int_0^T Y\big(y_0(t, c_0), y_0'(t, c_0), y_0''(t, c_0), t, 0\big)\,dt = 0$$

has a simple

$$B_0 := F'(c_0^*) \neq 0$$

real root $c_0^* \in \mathbb{R}^1$. To find a periodic solution of the Rayleigh type equation (10), we can use the Adomian decomposition method [9 – 12]. The periodic solution of the Rayleigh-type equation (10) in the critical case is given by

$$y(t, \varepsilon) := y_0\big(t, c_0^*\big) + u_1(t, \varepsilon) + \ldots + u_k(t, \varepsilon) + \ldots .$$

The nonlinear vector function $Y(z, z', t, \varepsilon)$ is analytical in terms of the unknown $y$ and its derivatives $y'$ and $y''$ in the neighborhood of the solution of the generating problem, so in this neighborhood there is a decomposition [9, p. 502]

$$\begin{aligned} Y(y(t,\varepsilon), y'(t,\varepsilon), t, \varepsilon) = {} & A_0\big(y_0\big(t, c_0^*\big)\big) + A_1\big(y_0\big(t, c_0^*\big), u_1(t,\varepsilon), \varepsilon\big) \\ & + A_2\big(y_0\big(t, c_0^*\big), u_1(t,\varepsilon), u_2(t,\varepsilon), \varepsilon\big) + \ldots \\ & + A_n\big(y_0\big(t, c_0^*\big), u_1(t,\varepsilon), \ldots, u_n(t,\varepsilon), \varepsilon\big) + \ldots . \end{aligned}$$

In the small neighborhood of the generating solution $z_0\big(t, c_r^*\big)$ there is also a decomposition

$$\begin{aligned} &Y\big(y_0\big(t, c_r^*\big) + x(t,\varepsilon), y_0'\big(t, c_r^*\big) + x'(t,\varepsilon), y_0''\big(t, c_r^*\big) + x''(t,\varepsilon), t, \varepsilon\big) \\ &\quad = Y\big(y_0\big(t, c_r^*\big), y_0'\big(t, c_r^*\big), y_0''\big(t, c_r^*\big), t, 0\big) + \mathfrak{A}_1(t)x(t,\varepsilon) + \mathfrak{A}_2(t)x'(t,\varepsilon) \\ &\qquad + \mathfrak{A}_3(t)x''(t,\varepsilon) + \varepsilon\,\mathfrak{A}_4(t) + \mathfrak{R}\big(y_0\big(t, c_r^*\big) + x(t,\varepsilon), y_0'\big(t, c_r^*\big) + x'(t,\varepsilon), t, \varepsilon\big), \end{aligned}$$

where

$$\mathfrak{A}_1(t) := \frac{\partial Y(y,y',y'',t,\varepsilon)}{\partial y}\bigg|_{\substack{y=y_0(t,c_r^*),\\ y'=y_0'(t,c_r^*),\\ y''=y_0''(t,c_r^*),\\ \varepsilon=0,}} \qquad \mathfrak{A}_2(t) := \frac{\partial Y(y,y',y'',t,\varepsilon)}{\partial y'}\bigg|_{\substack{y=y_0(t,c_r^*),\\ y'=y_0'(t,c_r^*),\\ y'=y_0''(t,c_r^*),\\ \varepsilon=0,}}$$

$$\mathfrak{A}_3(t) := \frac{\partial Y(y,y',y'',t,\varepsilon)}{\partial y''}\bigg|_{\substack{y=y_0(t,c_r^*),\\ y'=y_0'(t,c_r^*),\\ y''=y_0''(t,c_r^*),\\ \varepsilon=0,}} \qquad \mathfrak{A}_4(t) := \frac{\partial Y(y,y',y'',t,\varepsilon)}{\partial \varepsilon}\bigg|_{\substack{y=y_0(t,c_r^*),\\ y'=y_0'(t,c_r^*),\\ y''=y_0''(t,c_r^*),\\ \varepsilon=0.}}$$

Denote the constant matrix

$$\mathfrak{B}_0 := \ell K\big[\mathfrak{A}_1(s)X_r(s) + \mathfrak{A}_2(s)X_r'(s) + \mathfrak{A}_3(s)X_r''(s)\big](\cdot).$$

In the case of a periodic problem for a Rayleigh-type equation (10), the requirement (6) takes the form

$$\mathfrak{B}_0 \neq 0,$$

while equation (7) is uniquely solvable. In the case of (6), as in the Theorem, the sequence of approximations to the solution of a periodic problem for a Rayleigh-type equation (10) unsolved with respect to the derivative in a small neighborhood of the solution of the generating problem is determined by the iterative scheme

$$\begin{gathered} y_1(t,\varepsilon) := z_0\big(t, c_r^*\big) + u_1(t,\varepsilon), \qquad c_1 = -B_0^{-1}\, d_1, \\ u_1(t,\varepsilon) = X_r(t)c_0 + \varepsilon\, G[A_0(y_0(s, c_0^*))](t), \ldots, \\ y_{k+1}(t,\varepsilon) := y_0\big(t, c_0^*\big) + u_1(t,\varepsilon) + \ldots + u_{k+1}(t,\varepsilon), \\ y_{k+1}(t,\varepsilon) = X_r(t)c_{k+1} + \varepsilon\, G\big[A_k\big(y_0\big(s, c_r^*\big), u_1(s,\varepsilon), \ldots, u_k(s,\varepsilon)\big)\big](t), \\ c_k = -B_0^{-1}\, d_k, \quad k = 0, 1, 2, \ldots . \end{gathered} \tag{11}$$

The interval of values of the small parameter $\varepsilon \in [0, \varepsilon_0],\ 0 \leq \varepsilon_* \leq \varepsilon_0$, for which the iterative scheme (11) converges to the solution of the periodic problem for the Rayleigh-type equation (10) unsolved with respect to the derivative, can be evaluated similarly to [1, 11, 15 – 17]. Thus, the following statement is proved.

***Corollary.*** *For a periodic problem for a Rayleigh-type equation (10) unsolved with respect to the derivative, there is a critical* $(P_{Q^*} \neq 0)$ *case. Suppose that the condition (4) of solvability of the generating problem (3) is satisfied. Let us also assume that the equation (5) does not turn into an identity and has real solutions. Then for each root* $c_0^* \in \mathbb{R}^1$ *of equation (5), under the condition (6), the periodic problem for the Rayleigh-type equation (10) unsolved with respect to* *the derivative has a single analytical solution, which, for* $\varepsilon = 0,$ *transforms into a generating solution*

$$y_0\big(t, c_0^*\big) = X_r(t)c_0^* + g[f(s)](t).$$

*This solution can be determined, by using the iterative process (11). If there exists a constant* $0 < \gamma < 1,$ *for* $\varepsilon \in \big[0, \varepsilon^*\big]$ *such that the inequalities*

$$\|u_1(t,\varepsilon)\| \le \gamma \big\|y_0\big(t, c_0^*\big)\big\|, \qquad \|u_{k+1}(t,\varepsilon)\| \le \gamma \|u_k(t,\varepsilon)\|, \quad k = 1, 2, \ldots,$$

*hold, then the iterative scheme (11) converges to the solution of a periodic problem for a Rayleigh-type equation (10) unsolved with respect to the derivative.*

Let us apply the found conditions of solvability of the $T$-periodic boundary-value problem for the Rayleigh-type equation (10) unsolved with respect to the derivative to the $2\pi$-periodic boundary-value problem for the equation that determines the satellite motion [2].

***Example.*** The conditions of the proved corollary are satisfied in the case of a $2\pi$-periodic boundary-value problem for the equation that determines the motion of a satellite in an elliptical orbit

$$y'' = \varepsilon\, Y\big(y, y', y'', t, \varepsilon\big), \tag{12}$$

where in particular

$$Y\big(y, y', y'', t, \varepsilon\big) := 4\sin t - \sin y + 2\, y' \sin t - y'' \cos t.$$

The equation of the generating amplitudes in the case of a $T$-periodic boundary-value problem for the Rayleigh-type equation (12) unsolved with respect to the derivative has simple

$$B_0 = -2\pi \ne 0$$

real root $c_0^* = 0.$ Periodic solutions of the Rayleigh-type equation (12) unsolved with respect to the derivative will be sought in the neighborhood of the solution $y_0(t, c_0) \equiv 0$ of the linear part of this equation. The periodic problem for the first approximation equation is solvable due to the equality $F_0(c_0^*) = 0.$ By using the iterative scheme (11), we obtain

$$y_1(t,\varepsilon) := y_0\big(t, c_0^*\big) + u_1(t,\varepsilon), \qquad u_1(t,\varepsilon) = c_1 + \varepsilon\, g[A_0(y_0(s, c_0^*))](t) = -4\,\varepsilon \sin t.$$

Here,

$$A_0\big(y_0\big(t, c_0^*\big)\big) = 4\, \sin t, \qquad c_1 = 0.$$

In the second step, we obtain

$$y_2(t,\varepsilon) := y_0\big(t, c_0^*\big) + u_1(t,\varepsilon) + u_2(t,\varepsilon),$$

where

$$u_2(t,\varepsilon) = c_2 + \varepsilon\, g[A_1(y_0(s, c_0^*), u_1(s,\varepsilon))](t) = \varepsilon^2\, (3\cos t - 4) \sin t,$$

in addition

$$A_1\big(y_0\big(t, c_0^*\big), u_1(t,\varepsilon)\big) = 4\varepsilon\, (1 - 3\cos t) \sin t, \qquad c_2 = 0.$$

At the third step, we get

$$y_3(t,\varepsilon) := y_0\big(t, c_0^*\big) + u_1(t,\varepsilon) + u_2(t,\varepsilon) + u_3(t,\varepsilon),$$

where

$$u_3(t,\varepsilon) = c_3 + \varepsilon\, g[A_2(y_0(s,c_0^*),u_1(s,\varepsilon),u_2(s,\varepsilon))](t) = -\frac{\varepsilon^3}{24}\,(96\sin t - 45\sin 2t + 16\sin 3t),$$

in addition

$$A_2\big(y_0\big(t,c_0^*\big),u_1(t,\varepsilon),u_2(t,\varepsilon)\big) = \varepsilon^2\,(10 - 15\cos t + 12\cos 2t)\sin t, \qquad c_3 = 0.$$

At the fourth step, we have

$$y_4(t,\varepsilon) := y_0\big(t,c_0^*\big) + u_1(t,\varepsilon) + u_2(t,\varepsilon) + u_3(t,\varepsilon) + u_4(t,\varepsilon),$$

where

$$u_4(t,\varepsilon) = c_4 + \varepsilon\, g[A_3(y_0(s,c_0^*),u_1(s,\varepsilon),u_2(s,\varepsilon),u_3(s,\varepsilon))](t)$$
$$= -\frac{\varepsilon^4}{864}\,(3456\sin t + 1917\sin 2t - 1040\sin 3t + 270\sin 4t), \qquad c_4 = 0,$$

in addition

$$A_3\big(y_0\big(t,c_0^*\big),u_1(t,\varepsilon),u_2(t,\varepsilon),u_3(t,\varepsilon)\big) = -\frac{\varepsilon^3}{12}\,(-82 + 333\cos t - 260\cos 2t + 120\cos 3t)\sin t.$$

For the approximations to the periodic solution, found by using the iterative scheme (11) of the Rayleigh-type equation (12) unsolved with respect to the derivative for $\varepsilon_0 = 0{,}1,$ the inequalities hold

$$\|u_{k+1}(t,\varepsilon)\|_{\mathbb{C}[0,2\pi]} \le \gamma\|u_k(t,\varepsilon)\|_{\mathbb{C}[0,2\pi]}, \quad \gamma \approx 0{,}119\,443 \ll 1, \quad k = 1,2,3,$$

therefore, we can talk about the practical convergence of the iterative scheme (11) to a periodic solution of the Rayleigh-type equation (12) unsolved with respect to the derivative. Since for $\varepsilon \in [0;0{,}75]$ the following inequalities hold:

$$\|u_{k+1}(t,\varepsilon)\|_{\mathbb{C}[0,2\pi]} \le \gamma\|u_k(t,\varepsilon)\|_{\mathbb{C}[0,2\pi]}, \quad \gamma \approx 0{,}895\,822 < 1, \quad k = 1,2,3,$$

the interval of values of the small parameter $\varepsilon \in [0,\varepsilon_0],\ 0 \le \varepsilon_* \le \varepsilon_0,$ for which the convergence of the iterative scheme (11) to the solution of the periodic problem for the Rayleigh-type equation (10) is preserved, can be estimated as $\varepsilon \in [0, 0{,}75].$

The accuracy of the approximations to the periodic solution of the Rayleigh-type equations (12), found by using the iterative scheme (11), is characterized by the residuals

$$\Delta_k(\varepsilon) := \big\|y_k''(t,\varepsilon) - \varepsilon\, Y(y_k(t,\varepsilon),y_k'(t,\varepsilon),y_k''(t,\varepsilon),t,\varepsilon)\big\|_{\mathbb{C}[0;2\pi]}, \quad k = 0,1,2,3,4.$$

In particular,

$$\Delta_0(0{,}1) \approx 0{,}4, \qquad \Delta_1(0{,}1) \approx 0{,}0892\,783, \qquad \Delta_2(0{,}1) \approx 0{,}0148\,584,$$
$$\Delta_3(0{,}1) \approx 0{,}00201\,539, \qquad \Delta_4(0{,}1) \approx 0{,}000\;613\,871,$$
$$\Delta_0(0{,}01) \approx 0{,}04, \qquad \Delta_1(0{,}01) \approx 0{,}000\,897\,698, \qquad \Delta_2(0{,}01) \approx 0{,}0000\,151\,976,$$
$$\Delta_3(0{,}01) \approx 2{,}08\;430 \times 10^{-7}, \qquad \Delta_4(0{,}01) \approx 5{,}52\;017 \times 10^{-9}.$$

We have obtained constructive necessary and sufficient conditions for solvability and a scheme for constructing solutions of the nonlinear boundary-value problem (1), (2) unsolved with respect to the derivative. An iterative scheme (8) was constructed to find a solution to this problem. In difference from the results of [5], to find approximations to the solutions of a nonlinear boundary-value problem unsolved with respect to the derivative, we use the Adomian decomposition method rather than the method of simple iterations.

The effectiveness of the found conditions of solvability and the scheme for constructing solutions of the nonlinear boundary-value problem (1), (2) is demonstrated on the example of a periodic boundary-value problem for the equation (12), which determines the motion of a satellite [2]. When constructing approximations to the periodic solution of the Rayleigh-type equation (12), the exact fulfillment of the conditions of solvability is ensured at each step, which guarantees the absence of age terms. By using the example of a periodic boundary-value problem for equation (12), we demonstrate a scheme for estimating the length of the interval of values of the small parameter $\varepsilon \in [0, \varepsilon_0],\ 0 \leq \varepsilon_* \leq \varepsilon_0$, for which the convergence of the iterative scheme (11) is preserved. The obtained results continue the study of various nonlinear boundary-value problems [1, 20].

The accuracy of the approximations to the solutions of the nonlinear boundary-value problem (1), (2), obtained by using the iterative scheme (11), can be improved by using the least squares method [21] and for matrix boundary-value problems [22 – 24].

On behalf of all authors, the corresponding author states that there is no conflict of interest. All necessary data are included into the paper. All authors contributed equally to this work.

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